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Testing the null of stationarity and cointegration in multiple time series

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The Ohio State University, 1994

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TESTING THE NULL OF STATIONARITY AND COINTEGRATION IN MULTIPLE TIME SERIES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree of Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Byung Chul Ahn, B.A., M.A.

* * * * *

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1994

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To my mother and late father

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This dissertation is for my mother, late farther, my wife and all my family members in Korea.

VITA

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CHAPTER I

Introduction

Since the work by Nelson and Plosser (1982), it has been generally accepted that most economic time series are integrated of order one. Nonstationarity is tested by conventional unit root test such as the ADF and Z_α tests. Existing tests, with some exceptions, are tests that use the unit root as the null hypothesis and stationarity as an alternative. One serious drawback of these conventional unit root tests, however, is their low power.

The fact that time series appear to be nonstationary creates difficulties in economic theory. For example, the random walk hypothesis about exchange rate series and price level implies economic theory has no explainatory power and often, the predictions based on theory are even worse than simple time series predictions. The most challenging approach is the concept of cointegration by Engle and Granger (1987). Cointegration implies that there is equilibrium relationship among the nonstationary variables. That is, we can formulate linear functions of the nonstationary time series that are stationary.

The conventional cointegration tests are straight forward extensions of the unit root tests using non-cointegration as the null hypothesis and cointegration as an alternative. That is, the conventional approach tests whether there is a unit root in the deviation from the equilibrium relationship.

Recently, the nonstationarity of macro economic time series has been debated in the presence of structural breaks. Perron (1989) suggests that most economic time series are, in fact, stationary around a broken trend. Accoring to Perron (1989), the null of nonstationarity is spuriously accepted by conventional unit root tests, in the presence of a structural break.

The purpose of this disertation is threefold. First, we propose various test statistics for the null of stationarity against the alternative of nonstationarity. Our test statistics are designed to be used for univariate as well as multivariate time series. The tests using the null hypothesis of stationarity are at least useful as a confirmatory data analysis tool. We derive the limiting distributions for various test statistics and investigate their finite sample properties by direct simulation. It turns out that our test statistics are reasonably powerful. In addition, we compare two testing strategies for multiple time series: applying univariate tests for each component of a multiple time series and multivariate tests. It is found that the latter is a better testing strategy in terms of finite sample size and power than the former in many cases. To assess the finite sample properties, we compare two methods of choosing lag length: fixed lag length and automatic lag selection.

Second, we propose cointegration tests that can be used for a single equation as well as a system of equations. We use cointegration as the null hypothesis and no-cointegration as an alternative hypothesis. Limiting distributions for the test statistics are derived and tabulated. To obtain nuisance parameter-free test statistics, the CCR (canonical cointegration regression) is used in cointegration regression. It is also shown that other efficient estimators such as FM-OLS could be used to obtain the same analytical results. Finite sample properties are investigated by direct simulation. Further, we compare different testing strategies for a system of equations. Specifically, we consider a system of two equations and two regressors.

Third, we suggest stationarity tests for multivariate time series allowing structural breaks. As argued by Perron (1989), structural break may cause spurious nonstationary. Therefore, conventional cointegration tests are inconsistent and tend to accept the null of nonstationarity. It is also true that stationarity tests are always divergent and tend to reject the null of stationarity when structural breaks are ignored. To construct consistent tests under the condition, we use stationarity with structural breaks as the null hypothesis against of nonstationarity as an alternative hypothesis. Test statistics are direct extensions of stationarity tests. We can allow a variety of structural breaks for which limiting distributions are derived and tabulated. Unlike Perron (1989), these test statistics do not require exogeneous structural breaks and also allow unknown structural break points.

CHAPTER II

Testing the Null of Stationarity for Multiple Time Series

2.1 Introduction

For univariate time series, there have been several different approaches to the problem of testing the null of stationarity against the alternative of unit root nonstationarity; see Park and Choi (1988), Bierens (1990), Herce (1989), DeJong, Nankervis, Savin and Whiteman (1992), Saikkonen and Luukkonen (1993a, 1993b), Arellano and Pantula (1990), Kwiatkowski, Phillips, Schmidt and Shin (1992, hereafter KPSS), Tanaka (1990), Khan and Ogaki (1992), Stock (1992), Tsay (1993) and Choi (1992b). In addition, Choi and Yu (1993) provide a general framework that generates many of these tests.

However, there has been no procedure available for testing the null of stationarity for multiple time series. As a result, researchers had to be content with applying the univariate tests to each element of a multiple time series in order to investigate the nature of the multiple time series. This procedure of applying the univariate tests multiple times is cumbersome and ignores the correlations among the elements of the multiple time series. Therefore, the purpose of this chapter is to introduce the tests for the null of stationarity that can be applied to multiple time series both with and without the presence of time trends. In order to generate the tests, we formulate the multivariate AR(1) representation for a given data by summing it and then apply the multivariate AR unit root tests [cf. Phillips and Durlauf (1986)]. This method generates various consistent tests in a unified manner, which generalize the univariate tests for the null of stationarity studied in Choi and Yu (1993). We also report extensive simulation results that check the finite sample performance of the tests. In particular, we will compare the strategy of applying the univariate tests several times and that of using the multivariate tests. The experimental results in Section 5 indicate that there are merits in using the multivariate tests rather than applying the univariate tests several times.

This chapter is organized as follows. Section 2 introduces the test statistics. Section 3 derives the limiting distributions of the tests for general time series. Section 4 extends the tests in Section 3 to the case where time trends are present. Section 5 and 6 report simulation results. Tests are applied to the real interest rate data in Section 7. Section 8 concludes with a summary and further remarks. All proofs are in the Appendix A.

A few words on our notation: All the limits are taken as "$T \rightarrow \infty$" unless otherwise specified. Weak convergence is denoted as "$\Rightarrow$". Additionally, "Δ" signifies the usual difference operator. The standard n-vector Brownian motion is written as "$W(r)$" and "$f_{vv}(\cdot)$" denotes the spectral density matrix for $\{v_t\}$. The indicator function is signified as "$\iota(\cdot)$". Letting the matrix $A = [a_1, a_2, \cdots, a_n]'$, $vec(A) = [a_1', a_2', \cdots, a_n']'$, Last, "$A^{(i,j)}$" denotes the $(i, j) - th$ element of the matrix A.

2.2 Test Statistics

Let $\{x_t\}_{t=1}^{T}$ be an n-vector time series defined on the probability space (X, F, P). We are interested in testing the null hypothesis

$$H_0 : x_t = I(0) \tag{2.1}$$

against the alternative

$$H_1 : x_t^{(i)} = I(k_i), k_i \geq 1 \text{ for some } i \tag{2.2}$$

[see Engle and Granger (1987) for the definition of $I(k)$]. Under the alternative, we allow each element of x_t to have different order of integration but require that at least one element be nonstationary. Examples of x_t under the alternative are:

$$x_t = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} x_{t-1} + u_t, |\alpha| < 1, u_t = I(0) \tag{2.3}$$

and

$$x_t = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} x_{t-1} + u_t, |\alpha| < 1, u_t = I(0). \tag{2.4}$$

For (2.3), $x_t^{(1)} = I(1)$ and $x_t^{(2)} = I(0)$. For (2.4), $x_t^{(1)} = I(1)$ and $x_t^{(2)} = I(2)$ when $\alpha \neq 0$; $x_t^{(1)}$, $x_t^{(2)} = I(1)$ when $\alpha = 0$. We also allow the nonstationary elements of the time series x_t to be cointegrated under the alternative. That is, letting $x_t = [s_t', z_t']'$ where the $s \times 1$ vector s_t is stationary and the $(n-s) \times 1$ vector z_t nonstationary, there may exist an $m \times (n-s)$ matrix $C_1 (m < n-s)$, such that $C_1 z_t = [I(l_1), I(l_2), \cdots, I(l_m)]'$, $0 \leq l_j < \min(k_{s+1}, k_{s+2}, \cdots, k_n)$, $j = 1, \cdots, m$. This definition is slightly more general than Engle and Granger's (1987) original definition of cointegration; Engle and Granger assume that each element of x_t has the same order of integration.

There is no parameter of interest for the null (2.1), but we may artificially create the point null hypothesis by aggregating the time series $\{x_t\}$ as shown below. Since we have under the null, denoting $S_t = \sum_{i=1}^{T} x_i$ with $S_0 = 0$,

$$S_t = AS_{t-1} + x_t,\ A = I_n,\ (t = 1, 2, \cdots, T), \tag{2.5}$$

the null hypothesis (2.1) is equivalent to

$$H_0' : A = I_n \text{ and } x_t = I(0). \tag{2.6}$$

Most AR unit root tests are actually joint hypotheses tests for the location of the AR(1) coefficient and the assumption that the error terms are stationary. When the given assumption on the error terms is violated, most test statistics diverge. Therefore, the null of stationarity can be tested by the multivariate AR unit root tests for equation (2.5), assuming that $\{S_t\}$ is an observed time series. Notice that under the alternative we have $A = I_n$, yet at least one element of x_t is nonstationary, and hence the tests we will propose diverge in probability under the alternative, yielding consistent tests.

To derive the *LM* tests for the null hypothesis (2.6), we assume $x_t \sim iid\ N(0, \Omega)$, where Ω is a positive definite matrix. The log-likelihood function for equation (2.5) is written as

$$L(A, \Omega) = -nT/2 - (T/2)ln|\Omega| - \frac{1}{2}\sum_{1}^{T} tr\{\Omega^{-1}(S_t - AS_{t-1})(S_t - AS_{t-1})'. \tag{2.7}$$

Therefore, under the null hypothesis,

$$\partial L(A, \Omega)/\partial vec(A) = (\Omega^{-1} \otimes I_n)vec(\sum_{t=2}^{T} \Delta S_t S_{t-1}') \tag{2.8}$$

and

$$\partial^2 L(A,\Omega)/\partial vec(A)^2 = -(\Omega^{-1} \otimes \sum_{t=2}^{T} S_{t-1}S'_{t-1}). \qquad (2.9)$$

Because the information matrix is block-diagonal, we formulate the *LM* tests for the null (2.6) as follows:

$$\begin{aligned} LM_I &= \{vec(\sum_{t=2}^{T} \Delta S_t S'_{t-1})\}'(\hat{\Omega}^{-1} \otimes T^{-2}\hat{\Omega}^{-1}) vec(\sum_{t=2}^{T} \Delta S_t S'_{t-1}) \\ &= tr\{(T^{-1}\sum_{t=2}^{T} \Delta S_t S'_{t-1})\hat{\Omega}^{-1}(T^{-1}\sum_{t=2}^{T} S_{t-1}\Delta S'_t)\hat{\Omega}^{-1}\} \qquad (2.10) \end{aligned}$$

and

$$\begin{aligned} LM_{II} &= \{vec(\sum_{t=2}^{T} \Delta S_t S'_{t-1})\}'\{\hat{\Omega}^{-1} \otimes (\sum_{t=2}^{T} S_{t-1}S'_{t-1})^{-1}\} vec(\sum_{t=2}^{T} \Delta S_t S'_{t-1}) \\ &= tr\{(\sum_{t=2}^{T} \Delta S_t S'_{t-1})(\sum_{t=2}^{T} S_{t-1}S'_{t-1})^{-1}(\sum_{t=2}^{T} S_{t-1}\Delta S'_t)\hat{\Omega}^{-1}\}, \qquad (2.11) \end{aligned}$$

where $\hat{\Omega} = T^{-1}\sum_{t=1}^{T} \Delta S_t \Delta S'_t$ is a positive definite matrix and converges to Ω in probability. The difference between LM_I and LM_{II} lies in how the estimate of the information matrix is chosen. Notice that these two tests are invariant with respect to transformations $S^*_t = DS_t$ for nonsingular D. Also, these two tests reduce to the *LM* tests proposed in Choi and Yu (1993) when $n = 1$.

Rewriting (2.5) as

$$\Delta S_t = BS_{t-1} + x_t,\ B = 0,\ (t = 1, 2, \cdots, T), \qquad (2.12)$$

$T^{-1}LM_{II}$ is equivalent to the Bartlett-Nanda-Pillai trace test in multivariate analysis [cf. Anderson (1984, *p.* 334)]. The equivalence of the Bartlett-Nanda-Pillai trace test and the LM test has also been established elsewhere; see, for example, Anderson and Kunimoto (1992). In addition, replacing $\hat{\Omega}$ with $\Omega^* = T^{-1}\sum_{t=1}^{T}(\Delta S_t - \hat{B}S_{t-1})(\Delta S_t - \hat{B}S_{t-1})'$ where $\hat{B}' = (\sum_{t=1}^{T} S_{t-1}S_{t-1}')^{-1}(\sum_{t=1}^{T} S_{t-1}\Delta S_t')$, we find that $T^{-1}LM_{II}$ is equivalent to the Lawley-Hotelling trace test [cf. Anderson (1984, *p.* 334)], which in turn is equivalent to the Wald test upon standardization. Since $\hat{\Omega}$, $\Omega^* \xrightarrow{p} \Omega$ under the null, the LM_{II} and Wald tests have the same limiting distribution. We also deduce from the above and the inequality for the Wald, likelihood ratio and LM tests [cf. Berndt and Savin (1977)] that the likelihood ratio test has the same asymptotic distribution as LM_{II}. Because LM_{II} is more convenient to use and is likely to have virtually the same power properties as the Wald and likelihood ratio tests both in finite samples and asymptotically, we will not consider the latter two tests.

We may also consider a multivariate analog of the Sargan-Bhargava [cf. Sargan and Bhargava (1983)] and Durbin-Hausman [cf. Choi (1992c)] tests for an AR unit root:

$$SBDH = tr\{(T^{-2}\sum_{t=1}^{T} S_t S_t')\hat{\Omega}^{-1}\}. \tag{2.13}$$

This test is invariant with respect to nonsingular linear transformations as the LM_I and LM_{II} tests.

Assuming that S_t is not cointegrated, we deduce the asymptotic distributions of these tests easily from Phillips and Durlauf (1986). These are:

$$LM_I \Rightarrow tr\{\int_0^1 dW(r)W(r)' \int_0^1 W(r)dW(r)'\}, \tag{2.14}$$

$$LM_{II} \Rightarrow tr[\int_0^1 dW(r)W(r)'\{\int_0^1 W(r)W(r)'dr\}^{-1}\int_0^1 W(r)dW(r)'], \tag{2.15}$$

$$SBDH \Rightarrow tr\{\int_0^1 W(r)W(r)'dr\}. \tag{2.16}$$

Notice that LM_{II} has the same asymptotic distribution as Johansen's (1988) test for the number of cointegrating vectors. Under the alternative, all the test statistics diverge to infinity in probability. This will be discussed later in more general contexts.

2.3 Extensions to General DGPS

In this section, we extend the tests in Section 2 to more general data generating processes (DGPs). Namely, we assume that $x_t = v_t$ under the null and that $\Delta^{k_i}x_t^{(i)} = v_t^{(i)}$ under the alternative, where $\{v_t\}$ is a vector linear process. More specifically, we make the following assumptions regarding $\{v_t\}$:

$$\begin{aligned}
A1 &: v_t = \sum_{i=0}^{\infty} C_i e_{t-i} \\
A2 &: \sum_{i=0}^{\infty} i\|C_i\| < \infty \\
A3 &: \sum_{i=0}^{\infty} C_i \neq 0 \\
A4 &: \{e_t, F_t\} \text{ is a vector martingale difference sequence,} \\
A5 &: E(e_t e_t'|F_{t-1}) = \Sigma,\ \Sigma \text{ is positive definite,} \\
A6 &: \sup_{i,t} E(|e_t^{(i)}|^{2+\delta}|F_{t-1}) < \infty \text{ for some } \delta > 0, \\
A7 &: \Omega_l = 2\pi f_{vv}(0) = (\sum_{i=0}^{\infty} C_i)\Sigma(\sum_{i=0}^{\infty} C_i)' \text{ is positive definite,}
\end{aligned}$$

where $C_i's$ are real matrices and $\|C_i\| = \{tr(C_i'C_i)\}^{1/2}$. A stationary and invertible vector ARMA process is a special case of $\{v_t\}$. Under A1, A2, A4, A5 and A6, we have as in Phillips and Solo (1992, *p.* 985)

$$\Omega_l^{-1/2}T^{-1/2}\sum_{t=1}^{[Tr]} v_t \Rightarrow W(r). \tag{2.17}$$

Also, extending Hannan and Heyde's (1972) results, we have under A1, A4, A5 and an assumption implied by A2 that

$$T^{-1}\sum_{t=1}^{T} v_t v_t' \xrightarrow{p} \Omega_0, \tag{2.18}$$

where $\Omega_0 = E(v_t v_t') = \sum_{i=0}^{\infty} C_i \Sigma C_i'$. A3 is required to ensure that the limiting distribution of the partial sum process in (2.17) is non-degenerate and to ensure that $\{v_t\}$ does not have an MA unit root. A7 implies that S_t is not cointegrated under the null hypothesis.

We modify the *LM* and *SBDH* tests in Section 2 along the lines of Phillips (1987) and Phillips and Durlauf (1986) such that the asymptotic distributions of these tests are free of nuisance parameters. The modified test statistics are defined as follows:

$$LM_I^m = tr\{(T^{-1}\sum_{t=2}^{T}\Delta S_t S_{t-1}' - \hat{\Omega}_1')\hat{\Omega}_l^{-1}(T^{-1}\sum_{t=2}^{T} S_{t-1}\Delta S_t' - \hat{\Omega}_1)\hat{\Omega}_l^{-1}\}, \tag{2.19}$$

$$LM_{II}^m = tr\{(\sum_{t=2}^{T}\Delta S_t S_{t-1}' - T\hat{\Omega}_1')(\sum_{t=2}^{T} S_{t-1}S_{t-1}')^{-1}(\sum_{t=2}^{T} S_{t-1}\Delta S_t' - T\hat{\Omega}_1)\hat{\Omega}_l^{-1}\} \tag{2.20}$$

and

$$SBDH^m = tr\{(T^{-2}\sum_{t=1}^{T} S_t S_t')\hat{\Omega}_l^{-1}\}, \tag{2.21}$$

where $\hat{\Omega}_l$ and $\hat{\Omega}_1$ are consistent estimates of Ω_l and $\Omega_1 = \sum_{t=2}^{\infty} E(v_1 v_t')$, respectively. As in Hannan (1970), we define $\hat{\Omega}_l$ as

$$\hat{\Omega}_l = \sum_{n=-l}^{l} \hat{C}(n)k(n/l), \tag{2.22}$$

$$\hat{C}(n) = T^{-1} \sum_{t=2}^{T-n} \Delta S_t \Delta S'_{t+n} \tag{2.23}$$

and $k(n/l)$ is a lag window. Analogously, we define $\hat{\Omega}_1 = \sum_{n=1}^{l} \hat{C}(n)k(n/l)$.

Regarding the lag truncation number l and the spectral window, we assume

$$A8 : l \to \infty \text{ as } T \to \infty \text{ and } l = O(T^{\delta}),\ 0 < \delta < \frac{1}{2}.$$

This ensures that the spectral density estimates are consistent. Further, we assume for the lag window $k(z)$ that

$$\begin{aligned} A9 \quad : \quad & k(z) \text{ is a continuous, even function with} \\ & k(0) = 1,\ |k(z)| < 1 \text{ and } \int_{-\infty}^{\infty} k^2(z)dz < \infty. \end{aligned}$$

Assumptions A8 and A9 imply that

$$\sum_{n=-l}^{l} k(n/l) = O(T^{\delta}) \text{ as } l, T \to \infty \text{ and } 0 < \delta < \frac{1}{2}.$$

This result will be used to derive the rate of divergence of the test statistics under the alternative and under the null with misspecified time trends.

Asymptotic properties of the tests proposed in this section are reported in the following theorem:

Theorem 1: *Suppose that assumptions A1-A9 hold.*

(a) Under the null hypothesis (2.1),

$$(i)\ LM_I^m \Rightarrow tr\{\int_0^1 dW(r)W(r)' \int_0^1 W(r)dW(r)'\}, \tag{2.24}$$

$$(ii)\ LM_{II}^m \Rightarrow tr[\int_0^1 dW(r)W(r)'\{\int_0^1 W(r)W(r)'dr\}^{-1} \int_0^1 W(r)dW(r)'], \tag{2.25}$$

$$(iii)\ SBDH^m \Rightarrow tr\{\int_0^1 W(r)W(r)'dr\}. \tag{2.26}$$

(b) Under the alternative hypothesis (2.2),

$$(i)\ LM_I^m = O_p(T^{2(1-\delta)}), \tag{2.27}$$

$$(ii)\ LM_{II}^m = O_p(T^{1-\delta}), \tag{2.28}$$

$$(iii)\ SBDH^m = O_p(T^{1-\delta}), \tag{2.29}$$

where $0 < \delta < \frac{1}{2}$.

Remarks:

(a) The tests studied in this theorem can be used exclusively for the multiple time series with zero means.

(b) The asymptotic distributions of the LM_{II}^m have been used to test the number of cointegrating vectors in Johansen (1988). To our knowledge, the distributions for the others have not been used for any statistical inference .

(c) We report the simulated percentiles of the asymptotic distributions up to $n =$ 6 in Part (a) of Table 1, which we obtained by generating independent n-vector standard normal variates for $\{v_t\}_{t=1}^{500}$ 100,000 times except for LM_I^m under $n = 1$. The percentiles for LM_I^m under $n = 1$ were taken from Choi (1992a), which reports its exact asymptotic pdf and cdf. Note also that the asymptotic distribution of LM_{II}^m is tabulated in Johansen (1988) by simulation. Our simulated percentiles are almost identical with those reported in Johansen. We observe that the asymptotic distributions shift away from the origin as we add more variables to the system. We infer from this that the powers of the tests decrease as the number of variables in the system increases.

(d) In light of Theorem 1 (b), we reject the null hypothesis when the computed values of the test statistics are greater than the corresponding critical values.

2.4 Extensions to DGPS with time trends

We extend the tests in Section 3 to DGPs with time trends in this section. We assume that the observed vector-valued time series $\{y_t\}_{t=1}^T$ is generated under both the null and alternative by

$$\Delta^{k_i}\{y_t^{(i)} - \mu_0^{(i)} - \mu_1^{(i)}t - \cdots - \mu_{p_i}^{(i)}t^{p_i}\} = v_t^{(i)},\ \mu_{p_i}^{(i)} \neq 0,\ i = 1,\cdots,n, \tag{2.30}$$

where $\{v_t\}$ is a stationary vector linear process as in Section 3. Under the null hypothesis (2.1), $k_i = 0$ for all i; under the alternative $k_i \geq 1$ at least for one i. We assume that the true time polynomial orders p_i for $y_t^{(i)}$ are known and that they do not depend on the order of integration. This is not a restrictive assumption at least for economic time series, because it appears that $p_i = 0$ or 1 is adequate for most economic time series. The DGP (2.30) is equivalent to

$$y_t^{(i)} = \delta_{k_ik_i}^{(i)}t^{k_i} + \cdots + \delta_{k_ip_i}^{(i)}t^{p_i} + x_t^{(i)},\ (k_i \leq p_i),\ \delta_{k_ip_i}^{(i)} \neq 0, \tag{2.31}$$

$$y_t^{(i)} = x_t^{(i)},\ (k_i > p_i), \tag{2.32}$$

where $\Delta^{k_i}x_t^{(i)} = v_t^{(i)}[i.e., x_t^{(i)} = I(k_i)]$. As discussed in Choi and Yu (1993), this DGP is general enough to include most DGPs assumed in economic time series analyses. The DGP (2.32) was considered in Section 3 and, therefore, we assume the DGP

(2.31) in this section. We also allow (as in Section 2) that the nonstationary elements of $[x_t^{(1)}, \cdots, x_t^{(n)}]$ are cointegrated.

Moreover, we do not observe $S_t = \sum_{i=1}^t x_i$ in the DGP (2.31), which is required to formulate the tests for the null hypothesis (2.1). We can estimate them consistently under the null in two different ways. First, we may run the OLS regression

$$y_t = \bar{\delta}_0 t^0 + \cdots + \bar{\delta}_p t^p + \bar{x}_t,\ p = \max\{p_1, \cdots, p_n\}, \tag{2.33}$$

and let $\bar{S}_t = \sum_{i=1}^t \bar{x}_i$. Because $\bar{\delta}_i \xrightarrow{p} \delta_i$, $(i = 0, \cdots, p)$, $\{\bar{S}_t\}$ can be used to formulate the *SBDH* test. However, using $\{\bar{S}_t\}$ results in degenerate asymptotic distributions for the *LM* tests, because $\sum_{t=2}^T \Delta\bar{S}_t\bar{S}_{t-1}' = \frac{1}{2}(\bar{S}_T\bar{S}_T' - \sum_{t=1}^T \Delta\bar{S}_t\Delta\bar{S}_t')$ and $\bar{S}_T = 0$.

The second way of obtaining consistent estimates for $\{S_t\}$ is to run the OLS regression

$$P_t = \tilde{\delta}_0 \sum_{j=1}^t j^0 + \cdots + \tilde{\delta}_p \sum_{j=1}^t j^p + \tilde{S}_t,\ p = \max\{p_1, \cdots, p_n\}, \tag{2.34}$$

where $P_t = \sum_{j=1}^t y_j$. It is straightforward to show that $\tilde{S}_t \xrightarrow{p} S_t$ and $\Delta\tilde{S}_t \xrightarrow{p} x_t$ for all t. Because $\tilde{S}_T$ is not identically zero, we have well-defined asymptotic distributions for the *LM* tests when we use $\{\tilde{S}_t\}$. In addition, the *SBDH* tests can also be formulated by using $\{\tilde{S}_t\}$. As will be seen later, tests using $\{\bar{S}_t\}$ and $\{\tilde{S}_t\}$ have different asymptotic distributions.

We define the *LM* and *SBDH* tests in the same way as in Section 3.

$$LM_I^m = tr\{(T^{-1}\sum_{t=2}^T \Delta\tilde{S}_t\tilde{S}_{t-1}' - \tilde{\Omega}_1')\tilde{\Omega}_l^{-1}(T^{-1}\sum_{t=2}^T \tilde{S}_{t-1}\Delta\tilde{S}_t' - \tilde{\Omega}_1)\tilde{\Omega}_l^{-1}\}, \tag{2.35}$$

$$LM_{II}^m = tr\{(\sum_{t=2}^T \Delta\tilde{S}_t\tilde{S}_{t-1}' - T\tilde{\Omega}_1')(\sum_{t=2}^T \tilde{S}_{t-1}\tilde{S}_{t-1}')^{-1}(\sum_{t=2}^T \tilde{S}_{t-1}\Delta\tilde{S}_t' - T\tilde{\Omega}_1)\tilde{\Omega}_l^{-1}\}, \tag{2.36}$$

$$SBDH_T^m = tr\{(T^{-2}\sum_{t=1}^{T}\tilde{S}_t\tilde{S}_t')\tilde{\Omega}_l^{-1}\}, \tag{2.37}$$

and

$$SBDH_B^m = tr\{(T^{-2}\sum_{t=1}^{T}\bar{S}_t\bar{S}_t')\bar{\Omega}_l^{-1}\}, \tag{2.38}$$

where $\tilde{\Omega}_l$, $\bar{\Omega}_l$, $\tilde{\Omega}_1$ and $\bar{\Omega}_1$ are defined in the same way as in Section 3.

The asymptotic properties of the tests considered in this section are reported in the following theorem.

Theorem 2: *Suppose that assumptions A1-A9 hold.*

(a) Under the null hypothesis (2.1),

$$(i)\ LM_I^m \Rightarrow tr\{\int_0^1 d\tilde{W}(r)\tilde{W}(r)'\int_0^1 \tilde{W}(r)d\tilde{W}(r)'\}, \tag{2.39}$$

$$(ii)\ LM_{II}^m \Rightarrow tr[\int_0^1 d\tilde{W}(r)\tilde{W}(r)'\{\int_0^1 \tilde{W}(r)\tilde{W}(r)'dr\}^{-1}\int_0^1 \tilde{W}(r)d\tilde{W}(r)'], \tag{2.40}$$

$$(iii)\ SBDH_T^m \Rightarrow tr\{\int_0^1 \tilde{W}(r)\tilde{W}(r)'dr\}, \tag{2.41}$$

$$(iv)\ SBDH_B^m \Rightarrow tr\{\int_0^1 \bar{W}(r)\bar{W}(r)'dr\}, \tag{2.42}$$

where

$$\bar{W}(r) = W(r) - \bar{\alpha}_0 r^1/1 - \cdots - \bar{\alpha}_p r^{p+1}/(p+1), \tag{2.43}$$

$$\tilde{W}(r) = W(r) - \tilde{\gamma}_0 r^1/1 - \cdots - \tilde{\gamma}_p r^{p+1}/(p+1), \tag{2.44}$$

$\bar{\alpha}_i$ *and* $\tilde{\gamma}_i$ *minimize the least squares criteria in the* L_2 *norm, respectively,*

$$\int_0^1 \|W(r) - \alpha_0 r^0 - \cdots - \alpha_p r^p\|^2 dr, \tag{2.45}$$

$$\int_0^1 \|W(r) - \gamma_0 r^1/1 - \cdots - \gamma_p r^{p+1}/(p+1)\|^2 dr. \tag{2.46}$$

(b) Under the alternative hypothesis (2.2)

$$(i)\ LM_I^m = O_p(T^{2(1-\delta)}), \tag{2.47}$$

$$(ii)\ LM_{II}^m = O_p(T^{1-\delta}), \tag{2.48}$$

$$(iii)\ SBDH_B^m = O_p(T^{1-\delta}), \tag{2.49}$$

$$(iv)\ SBDH_T^m = O_p(T^{1-\delta}), \tag{2.50}$$

where $0 < \delta < \frac{1}{2}$.

Remarks:

(a) When $n = 1$, the asymptotic distributions derived in this theorem reduce to those of the tests studied in Choi and Yu (1993).

(b) Except for LM_I^m under $n = 1$, we tabulated the asymptotic percentiles of the tests in Theorem 2 by the same simulation methods as in Section 3 for the cases $p = 0$ or 1. The distributions for LM_I^m under $n = 1$ were taken from Choi (1992b) which reports the exact pdfs and cdfs of these tests. The percentiles are reported in parts (b) and (c) of Table 1. We observe as in Part (a) of Table 1 that the distributions shift away from the origin as the number of variables in the system increases. Therefore, the powers of the tests will decrease as the number of variables in the system increases.

(c) In light of Theorem 2 (b), we reject the null hypothesis when the computed values of the test statistics are greater than the corresponding critical values.

Also, it has been assumed that the true order of the time polynomial is known both under the null and alternative. But selecting order of time polynomial inappropriately may result in rejecting the null asymptotically when it is true. More specifically, assume that the true DGP is

$$y_t^{(i)} = \delta_{k_i k_i}^{(i)} t^{k_i} + \cdots + \delta_{k_i p_i}^{(i)} t^{p_i} + x_t^{(i)} \ (k_i \leq p_i),\ \delta_{k_i p_i}^{(i)} \neq 0,\ x_t = I(0), \tag{2.51}$$

but that the tests are formulated by using the regressions

$$y_t = \bar{\delta}_0 t^0 + \cdots + \bar{\delta}_q t^q + \bar{x}_t,\ q < \max\{p_1, \cdots, p_n\}, \tag{2.52}$$

$$P_t = \tilde{\delta}_0 \sum_{j=1}^{t} j^0 + \cdots + \tilde{\delta}_q \sum_{j=1}^{t} j^q + \tilde{S}_t,\ q < \max\{p_1, \cdots, p_n\}. \tag{2.53}$$

The behavior of the test statistics under this circumstance is analyzed in the following theorem.

Theorem 3: *Suppose that assumptions A1-A9 hold and that the time polynomial order q in the regression models (2.52) and (2.53) is chosen to be less than* $\max\{p_1, \cdots, p_n\}$ *where p_i denote the true time polynomial order for the $i-th$ element of y_t in (2.51). Then, under the null hypothesis (2.1),*

$$(i)\ LM_I^m = O_p(T^{2(1-\delta)}), \tag{2.54}$$

$$(ii)\ LM_{II}^m = O_p(T^{1-\delta}), \tag{2.55}$$

$$(iii)\ SBDH_T^m = O_p(T^{1-\delta}), \tag{2.56}$$

$$(iv)\ SBDH_B^m = O_p(T^{1-\delta}), \tag{2.57}$$

where $0 < \delta < \frac{1}{2}$.

Remark: These results indicate that we always reject the null asymptotically even when the null is true, if the true order of the time polynomial is underestimated. Obviously, we do not encounter such difficulties if the true order is overestimated. Therefore, in practice, it is advisable to make a generous choice of the order for regression time polynomials to avoid rejecting the null when it is true in fact. For most economic time series which show trend components, selecting $p_i = 1$ appears to be appropriate.

2.5 Finite Sample Power I

In this section, we investigate the finite sample performance of the tests introduced in Sections 3 and 4 by using simulation. In particular, we compare the testing strategy of applying univariate tests several times to each component of multiple time series with that of applying the multivariate tests to the series. The finite sample size and power of the tests proposed in Sections 3 and 4 depend on the sample size T, the lag length l for long-run variance estimation, the lag window chosen and the parameters associated with the DGP of $\{x_t\}$ [see Schmidt and Phillips (1992) for related analyses]. Further, the finite sample size and power may also depend on the initial variable x_0. But in this section, we used only the Bartlett lag window and chose $x_0 = 0$ for all the experimental results. The univariate and multivariate tests are expected to reject too often under the null as the initial variable takes larger values [cf. Choi (1992b)].

Random numbers for the simulation results were generated by the IMSL subroutine RNMVN. Empirical power was calculated out of 5,000 iterations at $T = 100$, 200, 400 by using the critical values reported in Table 1. For the long-run variance estimation, we chose three values of the lag length l, *i.e.*, $l = 2$, $l_1 = integer[4(T/100)^{1/4}]$ and $l_2 = integer[12(T/100)^{1/4}]$, following Schwert (1989). Note that $l_1 = 4$, 4 and 5 at $T = 100$, 200 and 400, respectively, and that $l_2 = 12$, 14 and 16 at $T = 100$, 200 and 400, respectively.

In Table 2, we report the empirical size of LM_I, LM_{II} and $SBDH$. Data were generated as

$$x_t = \begin{bmatrix} 0.8 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, x_0 = 0, e_t \sim iid\, N(0, \Omega), e_0 = 0, \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}. \quad (2.58)$$

Each component of the bivariate time series $\{x_t\}$ is $I(0)$; $\{x_t^{(1)}\}$ and $\{x_t^{(2)}\}$ are serially correlated. Note that the size of all the tests depends on the initial variable x_0 in finite samples. In Part (a), we report the size of the tests for the case where DGPs do not contain time trends. That is, $\{x_t\}$ is assumed to be the observed time series. The results for the univariate tests were obtained by calculating the fraction of replications for which the null of $I(0)$ is rejected for at least one series at the 5% level. Because the nominal frequency of non-rejection for the bivariate series is $0.95^2 = 0.9025$, the numbers for the univariate tests should be compared to $1 - 0.9025 \simeq 0.1$. When the numbers are greater than 0.1, the univariate tests are thought to reject too often under the null. For meaningful comparisons, we calculated the fraction of replications for which the multivariate tests reject the null at the 10% level. When we choose $l = l_2$, the size for the univariate and multivariate tests is close to 0.1 relative to other choices of the lag length but the univariate tests tend to reject slightly more often than the multivariate test. When $l = 2$ or l_1, the multivariate tests rejects more often. Further, we find that the LM_{II} tests tend to reject less often in both univariate and multivariate cases. In Part (b), the size of the tests for demeaned series is reported. We find again that $l = l_2$ yields size relatively close to 0.1 for both the univariate and multivariate tests. Comparing the univariate and multivariate tests, the univariate tests tend to reject slightly more often than the multivariate test when $l = l_2$. In most cases, the $SBDH$ tests tend to reject more often than the LM_I and LM_{II} tests, and LM_{II} tests tend to reject less often than the others. In Part (c), the size of the tests for demeaned and detrended series is reported. When $l = l_2$ is chosen, there are the least size distortions and the univariate tests tend to reject more often than the multivariate tests. Overall, the univariate $SBDH$ tests tend to reject more often than the other tests and the univariate LM_{II} test tends to reject less often than the other tests.

In Table 3, we report empirical power of LM_I, LM_{II} and $SBDH$. Data were generated as

$$x_t = \begin{bmatrix} 1.0 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, x_0 = 0, e_t \sim iid\, N(0, \Omega), e_0 = 0, \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}. \quad (2.59)$$

Note that $x_t^{(1)}, x_t^{(2)} = I(1)$ and that $\{x_t^{(1)}\}$ and $\{x_t^{(2)}\}$ are serially correlated. The finite sample power of all the tests does not depend on the initial variable x_0, excepting Part (a). In Part (a), we report the power of the tests for the case where DGPs do not contain time trends. The univariate and multivariate tests are reasonably powerful, but the multivariate tests are slightly more powerful than the univariate counterparts when $T = 200$ and 400. Among the multivariate tests, the LM_I tests appear to be more powerful than the others. In Part (b), the power of the tests for demeaned series is reported. We find that the multivariate tests are more powerful than the univariate counterparts. Among the multivariate tests, LM_{II} is the least powerful. In Part (c), the power of the tests for demeaned and detrended series is reported. The power of the test is lower than that for the demeaned series. Excepting the $SBDH$ tests, the multivariate tests reject more often than the univariate counterparts. The univariate

SBDH tests are more powerful than the multivariate counterparts, but remember that the univariate *SBDH* tests suffer from size distortions as we have seen in Part (*c*) of Table 2.

In Table 4, we report the empirical power of LM_I, LM_{II} and *SBDH* for the data generated by

$$x_t = \begin{bmatrix} 1.0 & 0.2 \\ 0.0 & 0.8 \end{bmatrix} x_{t-1} + e_t, x_0 = 0, e_t \sim iid\, N(0, \Omega), e_0 = 0, \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}. \quad (2.60)$$

Note that $x_t^{(1)} = I(1)$ and $x_t^{(2)} = I(0)$ and that $\{x_t^{(1)}\}$ and $\{x_t^{(2)}\}$ are serially correlated. The finite sample power of all the tests does not depend on the initial variable x_0, excepting Part (a). In Part (a), we report the power of the tests for the case where DGPs do not contain time trends. The univariate and multivariate tests are reasonably powerful, but the multivariate tests are overall more powerful than the univariate counterparts when $T = 200$ and 400. Comparing the multivariate tests, the LM_I test is most powerful at $T = 100$ but LM_{II} is most powerful at larger sample sizes. In Part (*b*), the power of the tests for demeaned series is reported. We find that the multivariate LM_I and LM_{II} tests are more powerful than the univariate counterparts but that the univariate *SBDH* tests are more powerful than the multivariate counterparts. Among the multivariate tests, LM_{II} is the least powerful. In Part (*c*), the power of the tests for demeaned and detrended series is reported. Excepting the *SBDH* tests, the multivariate tests reject more often than the univariate counterparts. Among the multivariate tests, the LM_I test is the most powerful.

To sum up our findings

(i) The multivariate tests show more stable size than their univariate counterparts when the lag length is chosen as $l = l_2$.

(ii) The multivariate tests are overall more powerful than their univariate counterparts. This is more conspicuously shown in the case of LM_I tests.

(ii) The multivariate LM_I tests show the most stable size and are the most powerful among the multivariate tests in most cases; therefore, the multivariate LM_I tests are preferred to other multivariate tests.

However, as is the case with most simulation studies, these conclusions depend on the experimental format chosen. Further, keep in mind that all the tests are not powerful for the demeaned and detrended series even at a sample size as large as $T = 200$. In addition, we may need to perform more simulation to characterize the finite sample performance of the tests more completely by varying the DGPs and the values of the initial variables.

2.6 Finite sample power II

In this section, we investigate the finite sample performance of the tests by using simulation. In particular, In addition, tests using the automatic lag selection and those using a fixed lag length are compared. In this section, we used only the quadratic spectral lag window and chose $x_0 = 0$ for all the experimental results.

In the unit root literature, size-adjusted empirical power is often reported, but the size-adjusted power for the stationarity tests does not provide meaningful information about the finite sample power of the tests because the empirical size of the stationarity tests depends on the value of the initial variable. For this reason, size adjustment was not made for the empirical power of the testes reported in this section.

Random numbers for the simulation results were generated by the IMSL subroutine RNMVN. The fixed lag truncation number for the long-run variance estimation was chosen as $l_2 = integer[12(T/100)^{1/4}]$, following Schwert (1989). Note that $l_2 = 12$, 14 and 16 at $T = 100$, 200 and 400, respectively. We obtained simulation results using $l_1 = integer[4(T/100)^{1/4}]$ (reported in previous section of this dissertation), but the results using l_2 appear to be more satisfactory. For the automatic lag selection, Andrews' (1991) methods with AR(4) and VAR(1) approximations for univariate and multivariate series, respectively, were used. We also tried the VAR(4) approximating model for multivariate series, but there were no significant differences. In order to make the tests consistent, we put a restriction that $\hat{l} = 2$ if $\hat{l} \geq T^{\epsilon}$. We chose $\epsilon = 0.7$ for raw series and $\epsilon = 0.65$ for detrended series.

In Table 34, we report the empirical size of LM_I, LM_{II} and $SBDH$. Data were generated as

$$x_t = \begin{bmatrix} 0.8 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, x_0 = 0, e_t \sim iid\, N(0, \Sigma), e_0 = 0, \Sigma = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}. \quad (2.61)$$

In Part (a), the results for the tests using raw series are reported. That is, $\{x_t\}$ is assumed to be the observed time series. When the automatic lag selection methods are used, the univariate tests are shown to keep the nominal size better than the multivariate counterparts, though both sets of tests show serious size distortions at $T = 100$. At $T = 200$ and $T = 400$, however, the univariate tests have empirical size reasonably close to 0.1. When the fixed lag is used, the univariate and multivariate tests show similar performance and the empirical size is reasonably close to 0.1 except LM_{II}. In addition, the tests using the fixed lag keep the nominal size appreciably better than those using the automatic lag selection. Comparing the three tests, LM_I and SBDH tend to reject more often than LM_{II} in all the cases.

In Part (b), the results based on demeaned series are reported. The multivariate tests using the automatic lag selection are shown to have appreciably better performance than those in Part (a). Especially at $T = 200$ and $T = 400$, the size is close to 0.1 except for LM_{II}, which tends to reject less often than the others. When the automatic lag is used, the multivariate tests show more stable size than the univariate counterparts except LM_I at $T = 100$. For both the univariate and multivariate tests, using the fixed lag yields empirical size reasonably close to 0.1 at all sample sizes, except for $SBDH_T$ at $T = 100$ and LM_{II}. Note also that the the univariate and multivariate tests show similar performance when the fixed lag is employed. Comparing the fixed and automatic lags for the multivariate tests, they provide almost similar results at $T = 200$ and $T = 400$. But at $T = 100$, the fixed lag yields better results than the automatic lag except for LM_{II}. For the univariate tests, the fixed lag yields better results at $T = 100$ and $T = 200$ except for LM_{II}. At $T = 400$, both the automatic and fixed lags yield similar results. Comparing the four tests, LM_I, $SBDH_T$ and $SBDH_B$ reject more often than LM_{II} in all the cases.

In Part (c), the results based on demeaned and detrended series are reported. We observe results similar to those in Part (b). The multivariate tests using the automatic lag selection keep the nominal size well at $T = 200$ and $T = 400$ except LM_{II} which tends to reject less often than the others; and perform slightly better than the univariate counterparts except LM_I at $T = 100$. Using the fixed lag yields the empirical size reasonably close to 0.1 for both the univariate and multivariate tests except for SBDH$_T$ at $T = 100$ and LM_{II}. The LM_{II} tests tend to reject too infrequently. Further, there are no appreciable differences between the univariate and multivariate tests when the fixed lag is used. For the multivariate tests, the fixed and automatic lags provide almost similar results at $T = 200$ and $T = 400$. But the fixed lag yields better results than the automatic lag except for LM_{II} at $T = 100$. For the univariate tests, the fixed lag yields slightly better results at $T = 100$ and $T = 200$ except for LM_I and LM_{II}. In addition, it is observed that the fixed and automatic lags yield similar results at $T = 400$. Comparison of the four tests yields the results similar to those in Part (b) : LM_I, SBDH$_T$ and SBDH$_B$ reject more often than LM_{II} in all the cases.

In Table 35, we report empirical power of LM_I, LM_{II} and $SBDH$. Data were generated as

$$x_t = \begin{bmatrix} 1.0 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, x_0 = 0, e_t \sim iid\, N(0, \Sigma), e_0 = 0, \Sigma = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}. \quad (2.62)$$

In Part (a), we report the power of the tests for raw time series. When the automatic lag is used, it is seen that the multivariate tests are less powerful than the univariate counterparts. Considering that the multivariate tests show more size distortions than the univariate counterparts, the power advantage of the univariate tests over the multivariate counterparts as we observe here appear to be real. Now comparing the univariate and multivariate tests using the fixed lag, we find that both sets of tests show similar performance except LM_{II} at $T = 100$ and $T = 200$. The power performance of the tests using the automatic lag is quite different from that of the tests using the fixed lag. In all the cases except the multivariate LM_{II} at $T = 200$, the tests using the automatic lag are more powerful than those using the fixed lag. This is well expected from the analytic result in Part (b) of Theorem 1. Comparing the three tests, LM_I and SBDH reject more often than LM_{II} in all the cases except those of the univariate tests using the fixed lag at $T = 200$ and $T = 400$.

In Part (b), the power of the tests for demeaned series is reported. Here we find that the multivariate tests using the automatic lag are more powerful than the univariate counterparts using the automatic lag unlike in Part (a). Because the multivariate tests keep the nominal size better than the univariate tests as we have seen in Part (2) of Table 2, these results imply that the multivariate tests using the automatic lag are more powerful than the univariate counterparts. Comparing the power performance of the tests using the fixed lag, we do not find any significant differences except that the multivariate LM_I is more powerful than the univariate LM_I. But there are quite striking differences in power performance between the tests using the automatic lag and those using the fixed lag. All the tests using the automatic lag are appreciably more powerful than corresponding tests using the fixed lag. In the case LM_I at $T = 200$, for example, the power gain for the univariate test is 0.76, while that for the multivariate LM_I is 0.64. Among the four tests we considered, LM_I, $SBDH_T$ and $SBDH_B$ appear to be more powerful than LM_{II}; and $SBDH_T$ and $SBDH_B$ are slightly more powerful than LM_I.

In Part (c), the results based on demeaned and detrended series are reported. In general, we find that the power of the tests decrease as compared to Part (b). For these results, we may give essentially the same interpretations as in Part (*b*). The multivariate tests are generally more powerful than the univariate counterparts; and the tests using the automatic lag are more powerful than the corresponding tests using the fixed lag without any exception. Further, LM_I, $SBDH_T$ and $SBDH_B$ appear to be more powerful than LM_{II}; and $SBDH_T$ and $SBDH_B$ are slightly more powerful than LM_I.

In Table 36, we report the empirical power of LM_I, LM_{II} and $SBDH$ for the data generated by

$$x_t = \begin{bmatrix} 1.0 & 0.2 \\ 0.0 & 0.8 \end{bmatrix} x_{t-1} + e_t, x_0 = 0, e_t \sim iid\, N(0, \Sigma), e_0 = 0, \Sigma = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}. \quad (2.63)$$

In Part (a), we report the power of the tests for raw time series. As in Part (a) of Table 3, the multivariate tests using the automatic lag are less powerful than the univariate counterparts in all the cases. When the fixed lag is used, the univariate tests are more powerful in most cases. Comparing the results based on the fixed and automatic lags, the tests using the automatic lag are more powerful in all the cases except the case of LM_{II} at $T = 200$ and $T = 400$. Comparing the four tests, LM_{II} tends to be less powerful than the others in most cases except a few.

In Part (*b*), the results for demeaned series are reported. Unlike in Part (a), the multivariate tests using the automatic lag are more powerful than the univariate counterparts using the automatic lag except $SBDH_T$ and $SBDH_B$. But these exceptional cases may be due to the size distortions of the univariate $SBDH_T$ and $SBDH_B$. When the fixed lag is used, the multivariate tests are more powerful than the univariate counterparts except $SBDH_T$ and $SBDH_B$ at $T = 100$. The power differences in these exceptional cases are minimal. In the case of LM_I, there are significant power advantages for the multivariate tests. At $T = 400$, for example, the multivariate test is twice as powerful as the univariate test. The power differences between the tests using the automatic and fixed lags are quite striking as in Part (*b*) of Table 3. These differences are more appreciable in larger samples, as we expect from Part (*b*) of Theorem 2. Among the four multivariate tests using the automatic lag, the LM_{II} test appears to be less powerful than the others.

In Part (*c*), the results for demeaned and detrended series are reported. As in Part (*c*) of Table 3, we find that the power of the tests decreases relative to Part (*b*). For these results, we may give essentially the same interpretations as in Part (*b*). The multivariate tests are generally more powerful than the univariate counterparts except $SBDH_T$ and $SBDH_B$ that use the automatic lag. But these exceptional results may be due to the size distortions of the univariate $SBDH_T$ and $SBDH_B$ tests using the automatic lag. Further, the tests using the automatic lag are more powerful than the corresponding tests using the fixed lag without any exception. Additionally, LM_I, $SBDH_T$ and $SBDH_B$ appear to be more powerful than LM_{II}.

To sum up our findings,

(i) For raw time series, the multivariate tests using the automatic lag selection methods do not keep the nominal size well relative to the univariate counterparts and are less powerful in most cases. But the univariate tests using the automatic lag keep the nominal size reasonably well in large samples and more powerful than the others. The univariate tests using the fixed lag keep the nominal size appreciably better than those using the automatic lag at $T = 100$ but are less powerful than the univariate tests using the automatic lag.

(ii) For detrended series, the multivariate tests using the automatic lag keep the nominal size reasonably well at $T = 200$ and $T = 400$ and outperform the univariate counterparts using the automatic lag selection in terms of size. Further, the multivariate tests using the automatic lag selection methods are appreciably more powerful than other kinds of tests. But at $T = 100$, the multivariate tests using the automatic lag selection, especially $SBDH_T$ and $SBDH_B$, suffer from size distortions.

Based on the various simulation results we have considered, we make the following recommendations for empirical practices.

(i) For zero-mean time series, use the univariate LM_I and $SBDH$ tests with the automatic lag selection when sample size is large. These tests are more powerful and keep the nominal size better than LM_{II}. When sample size is as small as 100, use the univariate LM_I and $SBDH$ tests with the fixed lag in order to minimize size distortions and attain high finite sample power.

(ii) For detrended time series, using the multivariate tests with the automatic lag selection appears to be a good testing strategy in the light of empirical size and power in moderately large samples. Especially, the LM_I, $SBDH_T$ and $SBDH_B$ tests are more powerful and keep the nominal size better than the LM_{II} test. But in samples as large as 100, size distortions are expected for the tests using the automatic lag. This is more conspicuous in the case of the $SBDH_T$ and $SBDH_B$ tests. Therefore, the results from the automatic multivariate tests at sample sizes as large as 100 should be interpreted with caution.

However, as is the case with most simulation studies, these recommendations critically depend on the experimental format chosen, and hence are at best of tentative nature. Therefore, the reader is advised to accept the above recommendations with this caveat in mind.

2.7 Applications

In this section, we apply the tests proposed in Section 4 to the data studied in Kugler and Neusser (1993). The data are the monthly real interest rates over the period 1980 to 1991 for the USA, Japan, the UK, the FRG, France and Switzerland. The reader is referred to Kugler and Neusser for more detailed descriptions of these data. Kugler and Neusser used the co-dependence approach due to Gourieroux and Peaucelle (1989) in order to test the real interest parity hypothesis. Because the co-dependence approach assumes that the given vector time series is stationary, Kugler and Neusser applied the unit root tests to each series and reported that the the null of a unit root is easily rejected for all the series. But the results from the augmented Dickey-Fuller tests seem to be sensitive to the choice of lag length, while the Phillips-Perron tests tend to give unanimous results. In order to check Kugler and Neusser's test results, we applied our tests to the series.

Before applying our tests to the Kugler-Neusser data, we drew the series in Figures 1-6. The figures show that there does not exist any noticeable trend component in each series. Therefore, we tested the null of level-stationarity (*i.e.*, $p_i = 0$ for $i = 1, 2, \cdots, 6$). We used the automatic lag selection method with the same lag window and restriction as in Section 6 for both the univariate and multivariate tests. First, we applied the univariate tests to each series, the results of which are reported in Part (a) of Table 37. It is shown that the null of level-stationarity is not rejected at the 5% significance level for all the series and tests, excepting the case of LM_I for France. But at the 10% level, the null is rejected for the USA series when the $SBDH_T$ and $SBDH_B$ tests are used. The results from applying the multivariate tests are reported in Part (*b*) of Table 37. Here we find that the null is not rejected at the 10% level for all the tests. The results in Table 37 and Figures 1-6 provide strong evidence that the real interest data are level-stationary, and hence support the unit-root test results reported in Kugler and Neusser. Further, it is illustrated that the multivariate tests which recognize the dynamic and static correlations among the six series provide more clear-cut evidence than the univariate tests.

2.8 Summary and Further Remarks

We have introduced tests for the null of stationarity that can be used for multiple time series. The asymptotic distributions were obtained in a unified manner by using the standard vector Brownian motion and the test consistency was established. The effects of misspecifying the order of time trends were also analyzed. Simulation results indicate that the tests we have introduced work reasonably well in finite samples and that using the multivariate tests is a better testing strategy than applying the univariate tests several times to each component of a multiple time series. The tests were applied to the real interest rate series of six major industrialized nations studied in Kugler and Neusser (1993). The multivariate tests are shown to provide clear-cut evidence that the vector time series are level-stationary. Among the multivariate tests we introduced, the LM_I tests show the best performance and are recommended for empirical work.

CHAPTER III

Testing for Cointergration in a System of Equations

3.1 Introduction

Since the influential work by Engle and Granger (1987), there have been many procedures for testing cointegration. Notably, most of the tests developed at the early stage of research in cointegration were designed for the null of non-cointegration (cf. Engle and Granger (1987), Phillips and Ouliaris (1988,1990), Johansen (1988), Stock and Watson (1988), Choi (1991,1992d)). However, it appears appropriate to take the null as cointegration, because most economic theories are based on long-run economic relationship or the cointegrating relation among economic variables and, therefore, it is desirable to minimize the error of falsely rejecting the null of cointegration. This point has been raised by many researchers whom we do not fully cite here.

In response to the need for cointegration tests that take the null as cointegration, there have been a few testing procedures most of which appeared relatively in recent years. These include Park (1990), Hansen (1992), Tanaka (1990), Shin (1993) and Quintos and Phillips (1992). Park (1990) uses the variable addition methods for devising tests, but the latter four articles use the framework of testing parameter constancy. Though all of these tests take the null as cointegration, the alternatives for these tests are different. The alternative for Park's and Shin's tests is explicitly non-cointegration. But Hansen's (except the L_c test) and Quintos and Phillips' tests do not take the alternative as non-cointegration, though these tests are expected to have asymptotic power against the alternative of non-cointegration. Further, all of the tests that appeared in the literature so far can be used only for a single equation. To date, there have not been any testing procedures that can be used for testing the null of cointegration in a system of equations. In the light of our general interest in simultaneous relations among economic variables, it is deemed useful to devise such procedures.

Therefore, the purpose of this chapter is to propose tests for the null of cointegration that can be applied to a system of equations. These tests are analogues to the tests for the null of stationarity for multiple time series studied in Chapter II. Unlike the previous approaches, we use a general framework that generates various kinds of consistent tests for the null of cointegration that have not been introduced in literature. This framework also generates Shin's (1993) tests when only a single equation is considered. Note that the same framework was also used for devising tests for the null of $I(m)$ against the alternative of $I(m+k)$ for univariate time series (cf. Choi and Yu (1993)).

In devising the cointegration tests for a system of equations, we use the residuals from Park's (1992) canonical cointegrating regression (CCR). This procedure has mainly been developed for efficient estimation of and statistical inference on cointegrating vectors. But in this chapter CCR is used to devise nuisance-parameter-free tests for cointegration. By contrast, it is difficult to eliminate nuisance parameters in the limit if we use OLS residuals to formulate the tests. This will be shown in Section 5. We may also use Phillips and Hansen's fully modified OLS (FM-OLS) methods instead of CCR, which is also illustrated in Section 5.

Using the multivariate tests for a system of equations is more convenient than applying univariate tests several times to each equation when we wish to test the cointegrating relations in more than one structural equations. Besides this convenience in use, some system cointegration tests have better finite sample properties than corresponding univariate tests as the simulation results in Section 6 indicate. In addition, once we establish cointegrating relations, we may use the CCR estimates already obtained for computing tests in order to do statistical inference on cointegrating matrices. These CCR estimates are also known to be efficient. Hence, estimation, statistical inference on cointegrating matrices and testing cointegration can be done simultaneously. This is in contrast to some other procedures in which testing cointegration and estimating cointegrating matrices are done separately.

This Chapter is organized as follows. Section 2 introduces the models, hypothesis and assumptions. Section 3 derives the asymptotic distributions for the multivariate feasible CCR estimates. Section 4 introduces test statistics and derives the asymptotic distributions of the tests for general time series. The rates of divergence of the tests under the alternative are also reported. Section 5 studies the properties of the tests based on residuals from other estimation methods (FM-OLS and OLS). Section 6 reports simulation results. Section 7 concludes with a summary and further remarks.

All proofs are in the Appendix B.

A few words on our notation: All the limits are taken as "$T \rightarrow \infty$" unless otherwise specified. Weak convergence is denoted as " $\Rightarrow$". Additionally, "Δ" signifies the usual difference operator. The relation of equivalence in distribution is denoted by " $\equiv$". When every element of the matrix A is $O_p(T^k)$, it is compactly written as "$A = O_p(T^k)$". Further, the spectral density matrix of the vector series $\{x_t\}$ is denoted as "$f_{xx}(\cdot)$". Last, "$A^{(i,j)}$" denotes the $(i,j)-th$ element of the matrix A.

3.2 The Models, Hypotheses and Assumptions

We consider the system of equations

$$y_t = Ax_t + u_t,\ (t = 1, 2, \cdots, T), \tag{3.1}$$

where y_t and x_t denote $n \times 1$ and $m \times 1$ vector time series, respectively. We assume that y_t, $x_t = I(1)$. When $u_t = I(0)$, each equation in the system of equations (3.1) signifies a cointegrating relation between an element of y_t and x_t. Methods of estimation and inference for the system of equations (3.1) are discussed in Phillips and Hansen (1990), Park (1992) and Phillips (1990), among others. When $u_t = I(1)$, the regression results based on the system of equations (3.1) are spurious in the sense of Granger and Newbold (1974), as analyzed in Phillips (1986).

As a direct extension of model (3.1), we may also consider

$$y_t = Hc_t + Ax_t + u_t,\ (t = 1, 2, \cdots, T), \tag{3.2}$$

where $c_t = [1,\ t, \cdots, t^p]'$ and $x_t = I(1)$. It is appropriate to use this model when y_t

contains nonstochastic time trends, as represented by

$$y_t = Fc_t + y_t^0,\ y_t^0 = I(1). \tag{3.3}$$

Asymptotic properties of the OLS estimates for this model up to $p = 1$ are studied in Park and Phillips (1988). In applications, the regressor x_t may also contain time trends. That is,

$$x_t = Gc_t + x_t^0,\ x_t^0 = I(1). \tag{3.4}$$

The asymptotic properties of the tests we are to introduce do not change at all for this regressor. This will be explained in a remark after Theorem 1 in Section 4. Therefore, we assume without loss of generality for the asymptotic properties of the tests that $G = 0$. However, when the regressor contains time trends, we need to estimate Δx_t^0 by running the OLS regression

$$x_t = \hat{G}c_t + \hat{x}_t^0, \tag{3.5}$$

or equivalently

$$\Delta x_t = \hat{K}\Delta c_t + \Delta\hat{x}_t^0 \tag{3.6}$$

in order to calculate CCR estimates.

We are interested in testing the null hypothesis

$$H_0 : u_t = I(0) \tag{3.7}$$

against the alternative

$$H_1 : u_t^{(i)} = I(1) \text{ for at least one } i. \tag{3.8}$$

The null hypothesis (3.7) is equivalent to that every equation in the system of equations (3.1) or (3.2) denotes the cointegrating relation, and hence that an equilibrium relation exists between y_t and x_t. Under the alternative, at least one element of u_t is nonstationary.

Letting $w_t = (u_t', \Delta x_t')'$, we assume under the null that w_t satisfies the assumptions A1-A9 in Chapter II. A stationary and invertible vector ARMA process is a special case of $\{w_t\}$. Under A1, A2, A4, A5 and A6, we have as in Phillips and Solo (1992, *p.* 985)

$$T^{-1/2}\sum_{t=1}^{[Tr]} w_t \Rightarrow B(r) = \begin{array}{c} \left[\begin{array}{c} B_1(r) \\ B_2(r) \end{array}\right] \end{array} \begin{array}{c} n \\ m \end{array}, \tag{3.9}$$

where $B(r)$ is a Brownian motion with covariance matrix Ω and $[x]$ denotes the integer part of x. Also, extending Hannan and Heyde's (1972) results, we have under A1, A4, A5 and an assumption implied by A2 that

$$T^{-1}\sum_{t=1}^{T} w_t w_t' \xrightarrow{p} \Lambda, \tag{3.10}$$

where $\Lambda = E(w_t w_t') = \sum_{i=0}^{\infty} C_i \Psi C_i'$. A3 is required to ensure that the limiting distribution of the partial sum process in (3.9) is non-degenerate and to ensure that $\{w_t\}$ does not have an MA unit root. A7 implies that $\sum_{i=1}^{t} w_i$ is not cointegrated under the null hypothesis.

Further, we decompose and partition Ω as

$$\Omega = \Lambda + \Sigma + \Sigma' = \begin{array}{cc} n & m \\ \end{array} \atop \left[\begin{array}{cc} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{array}\right] \begin{array}{c} n \\ m \end{array} \tag{3.11}$$

where $\Sigma = \sum_{i=1}^{\infty} E(w_t w_{t-i})$. Also, we let

$$\Gamma = \Lambda + \Sigma = \begin{array}{cc} \begin{array}{cc} n & m \end{array} & \\ \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} & \begin{array}{c} n \\ m \end{array} \end{array} \quad \begin{array}{cc} \begin{array}{cc} n & m \end{array} & \\ \begin{bmatrix} \Gamma_1 & \Gamma_2 \end{bmatrix} & n+m \end{array} \cdot \tag{3.12}$$

This partition of Γ will be used later for defining the feasible CCR estimators.

3.3 The Feasible CCR

In this section, we briefly explain the feasible CCR for multivariate time series. The reader is referred to Park (1992) and Park and Ogaki (1991) for details on the CCR methods. For the feasible CCR, we transform the regressor and regressand and then apply the OLS procedure. For the system of equations (3.1), the transformed model is given as

$$y_t^* = A x_t^* + u_t^*, \tag{3.13}$$

where

$$y_t^* = y_t - [\hat{\Lambda}^{-1}\hat{\Gamma}_2 \hat{A}' + (0, \hat{\Omega}_{12}\hat{\Omega}_{22}^{-1})']'\hat{w}_t, \tag{3.14}$$

$$x_t^* = x_t - (\hat{\Lambda}^{-1}\hat{\Gamma}_2)'\hat{w}_t, \tag{3.15}$$

$$u_t^* = u_t - \hat{\Omega}_{12}\hat{\Omega}_{22}^{-1}\Delta x_t - (\hat{A} - A)(\hat{\Lambda}^{-1}\hat{\Gamma}_2)'\hat{w}_t, \tag{3.16}$$

$$\hat{w}_t = (\hat{u}_t', \Delta x_t')', \tag{3.17}$$

$$\hat{u}_t = y_t - \hat{A} x_t, \tag{3.18}$$

$$\hat{A} = (\sum_{t=1}^{T} y_t x_t')(\sum_{t=1}^{T} x_t x_t')^{-1}, \tag{3.19}$$

$$\hat{\Lambda} = T^{-1}\sum_{t=1}^{T} \hat{w}_t \hat{w}_t', \tag{3.20}$$

$$\hat{\Omega} = \sum_{j=-l}^{l} \hat{C}(j)k(j/l), \tag{3.21}$$

$$\hat{C}(j) = T^{-1}\sum_{t=2}^{T-j} \hat{w}_t\hat{w}'_{t+j}, \tag{3.22}$$

$$\hat{\Sigma} = \sum_{j=1}^{l} \hat{C}(j)k(j/l), \tag{3.23}$$

and $k(\cdot)$ is a lag window. Note that $\hat{\Omega}$ and $\hat{\Gamma}$ are consistent estimates of Ω and Γ, respectively, and that $\hat{\Gamma}_2$, $\hat{\Omega}_{12}$ and $\hat{\Omega}_{22}$ are obtained from $\hat{\Gamma}$ and $\hat{\Omega}$ by appropriate partitions of these matrices.

The asymptotic distribution of the OLS estimate from the regression model (3.13) is given in the following lemma, which is a trivial extension of Park's (1992) Theorem 4.1 to the case of multivariate time series.

Lemma 1. *Suppose that assumptions A1-A9 hold true. Then, we have*

$$T(A^* - A) \Rightarrow \{\int_0^1 dB_{1.2}(r)B_2(r)'\}\{\int_0^1 B_2(r)B_2(r)'dr\}^{-1}, \tag{3.24}$$

where

$$A^* = (\sum_{t=1}^{T} y_t^* x_t^{*\prime})(\sum_{t=1}^{T} x_t^* x_t^{*\prime})^{-1} \tag{3.25}$$

and

$$B_{1.2}(r) = B_1(r) - \Omega_{12}\Omega_{22}^{-1}B_2(r). \tag{3.26}$$

Note that $B_{1.2}(r)$ is independent of $B_2(r)$ and that the covariance matrix of $B_{1.2}(r)$ is $\Omega_{11.2} = \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21}$.

For the system of equations (3.2), we run the OLS regression

$$y_t = \tilde{H}c_t + \tilde{A}x_t + \tilde{u}_t, \tag{3.27}$$

and let $\tilde{w}_t = (\tilde{u}_t', \Delta x_t')'$. Then, the transformed regressor and regressand are defined in the same way as for (3.13) except that we replace $\hat{A}$ and $\hat{w}_t$ with $\tilde{A}$ and $\tilde{w}_t$, respectively. When the regressor contains time trends as represented by equation (3.4), we let $\tilde{w}_t = (\tilde{u}_t', \Delta\hat{x}_t^{0\prime})'$, where $\Delta\hat{x}_t^0$ is from the OLS regression (3.5) or (3.6). The feasible CCR estimate of the coefficient matrix $B = (H, A)$ is obtained by running the OLS regression on

$$\bar{y}_t = B\bar{q}_t + \bar{u}_t, \tag{3.28}$$

where $\bar{q}_t = (c_t', \bar{x}_t')'$, and $\bar{y}_t$ and $\bar{x}_t$ denote the transformed time series. Note that we do not have to transform the time polynomials for the CCR. We report the asymptotic distribution of the OLS estimate of the coefficient matrix B in the following lemma.

Lemma 2. *Suppose that assumptions A1-A9 hold true. Then, we have*

$$(\bar{B} - B)D_T \Rightarrow \{\int_0^1 dB_{1.2}(r)Q(r)'\}\{\int_0^1 Q(r)Q(r)'dr\}^{-1}, \tag{3.29}$$

where

$$\bar{B} = (\sum_{t=1}^{T} \bar{y}_t\bar{q}_t')(\sum_{t=1}^{T} \bar{q}_t\bar{q}_t')^{-1}, \tag{3.30}$$

$$D_T = diag[T^{1/2}, T^{1+1/2}, \cdots, T^{p+1/2}, T, \cdots, T] \tag{3.31}$$

and

$$Q(r) = [1, r, \cdots, r^p, B_2(r)']'. \tag{3.32}$$

Remark: Note that this result is obtained for the case where the regressor x_t does not contain time trends. When the regressor x_t contains time trends as in equation (3.4) and, therefore, Δx_t^0 is estimated by an auxiliary OLS regression, some feasible CCR coefficient estimates for time trends do not have the same distributions as reported in this lemma. This problem is of separate interest and will be studied further elsewhere. It is also expected that other efficient procedures for estimating cointegrating equations with time trends share this problem. However, this problem does not cause any difficulties for the asymptotic distributions of the tests we are to derive in Section 4.

We use $\bar{S}_t = \sum_{i=1}^{t}(\bar{y}_i - \bar{B}\bar{q}_i)$ to formulate the *SBDH* test that will be introduced in Section 4. However, using $\bar{S}_t$ for the *LM* tests introduced in Section 4 results in degenerate asymptotic distributions, because $\sum_{t=2}^{T} \Delta\bar{S}_{t-1}\bar{S}'_{t-1} = \frac{1}{2}(\bar{S}_T\bar{S}_T - \sum_{t=1}^{T} \Delta\bar{S}_t\Delta\bar{S}'_t)$ and $\bar{S}_T = 0$.

Therefore, we consider the regression model

$$\bar{S}_t^y = B\bar{S}_t^q + \bar{S}_t^u, \tag{3.33}$$

where $\bar{S}_t^y = \sum_{i=1}^{t} \bar{y}_i$, $\bar{S}_t^q = \sum_{i=1}^{t} \bar{q}_i$ and $\bar{S}_t^u = \sum_{i=1}^{t} \bar{u}_i$. The OLS estimate of B from this regression model has the following asymptotic distribution.

Lemma 3. *Suppose that assumptions A1-A9 hold true. Then, we have*

$$(\check{B} - B)D_T \Rightarrow \{\int_0^1 B_{1.2}(r)S(r)'dr\}\{\int_0^1 S(r)S(r)'dr\}^{-1}, \tag{3.34}$$

where

$$\check{B} = (\sum_{t=1}^{T} \bar{S}_t^y \bar{S}_t^{q'})(\sum_{t=1}^{T} \bar{S}_t^q \bar{S}_t^{q'})^{-1},\ S(r) = \int_0^r Q(s)ds, \tag{3.35}$$

and D_T and $Q(s)$ are as defined in Lemma 2.

We will use the regression residual $\bar{S}_t^y - \check{B}\bar{S}_t^q$ in order to formulate the *LM* tests in Section 4.

3.4 Test Statistics and Asymptotic Results

We introduce tests for the null hypothesis (3.7) in this section. The tests are analogues of the multivariate tests for the null of stationarity introduced in Chapter II. Under the null, we have $u_t^* \xrightarrow{p} u_t - \Omega_{12}\Omega_{22}^{-1}\Delta x_t$ and $\bar{u}_t \xrightarrow{p} u_t - \Omega_{12}\Omega_{22}^{-1}\Delta x_t$ for each t. Because $u_t = I(0)$ under the null and $\Delta x_t = I(0)$ both under the null and alternative, the null hypothesis (3.7) is equivalent to, at least asymptotically,

$$H_0 : u_t^* = I(0) \text{ for model (3.1)} \tag{3.36}$$

and

$$H_0 : \bar{u}_t = I(0) \text{ for model (3.2).} \tag{3.37}$$

Because u_t^* and $\bar{u}_t$ are not observed in practice, we use CCR residuals to formulate the tests. The test statistics for the system of equations (3.1) are defined as follows:

$$LM_I = tr\{(T^{-1}\sum_{t=2}^{T}\Delta S_t^* S_{t-1}^{*\prime} - \hat{\kappa}\hat{\Sigma}'\hat{\kappa}')\Omega_{11.2}^{*-1}(T^{-1}\sum_{t=2}^{T}S_{t-1}^*\Delta S_t^{*\prime} - \hat{\kappa}\hat{\Sigma}\hat{\kappa}')\Omega_{11.2}^{*-1}\}, \tag{3.38}$$

$$LM_{II} = tr\{(\sum_{t=2}^{T}\Delta S_t^* S_{t-1}^{*\prime} - T\hat{\kappa}\hat{\Sigma}'\hat{\kappa}')(\sum_{t=2}^{T}S_{t-1}^* S_{t-1}^{*\prime})^{-1}(\sum_{t=2}^{T}S_{t-1}^*\Delta S_t^{*\prime} - T\hat{\kappa}\hat{\Sigma}\hat{\kappa}')\Omega_{11.2}^{*-1}\} \tag{3.39}$$

and

$$SBDH = tr\{(T^{-2}\sum_{t=1}^{T}S_t^* S_t^{*\prime})\Omega_{11.2}^{*-1}\}, \tag{3.40}$$

where

$$S_t^* = \sum_{i=1}^{t}(y_i^* - A^* x_i^*),\ \hat{\kappa} = [I,\ -\hat{\Omega}_{12}\hat{\Omega}_{22}^{-1}], \tag{3.41}$$

and

$$\Omega^*_{11.2} = \sum_{j=-l}^{l} D^*(j)k(j/l), \; D^*(j) = T^{-1}\sum_{t=1}^{T-j}(y^*_t - A^*x^*_t)(y^*_{t+j} - A^*x^*_{t+j})'. \tag{3.42}$$

Regarding the lag truncation number l for $\Omega^*_{11.2}$, we also assume A8 and A9. Note that $\Omega^*_{11.2}$ is a consistent estimate of $\Omega_{11.2} = \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21}$, which is the covariance matrix of $B_{1.2}(r)$.

We formulate the test statistics for the system of equations (3.2) by using $\bar{S}_t = \sum_{i=1}^{t}(\bar{y}_i - \bar{B}\bar{q}_i)$ from the regression equation (3.28) and $\check{S}_t = \bar{S}^y_t - \check{B}\bar{S}^q_t$ from the regression equation (3.33). The test statistics are

$$LM_I = tr\{(T^{-1}\sum_{t=2}^{T}\Delta\check{S}_t\check{S}'_{t-1} - \tilde{\kappa}\tilde{\Sigma}'\tilde{\kappa}')\check{\Omega}^{-1}_{11.2}(T^{-1}\sum_{t=2}^{T}\check{S}_{t-1}\Delta\check{S}'_t - \tilde{\kappa}\tilde{\Sigma}\tilde{\kappa}')\check{\Omega}^{-1}_{11.2}\}, \tag{3.43}$$

$$LM_{II} = tr\{(\sum_{t=2}^{T}\Delta\check{S}_t\check{S}'_{t-1} - T\tilde{\kappa}\tilde{\Sigma}'\tilde{\kappa}')(\sum_{t=2}^{T}\check{S}_{t-1}\check{S}'_{t-1})^{-1}(\sum_{t=2}^{T}\check{S}_{t-1}\Delta\check{S}'_t - T\tilde{\kappa}\tilde{\Sigma}\tilde{\kappa}')\check{\Omega}^{-1}_{11.2}\}, \tag{3.44}$$

$$SBDH_I = tr\{(T^{-2}\sum_{t=1}^{T}\check{S}_t\check{S}'_t)\check{\Omega}^{-1}_{11.2}\}, \tag{3.45}$$

$$SBDH_{II} = tr\{(T^{-2}\sum_{t=1}^{T}\bar{S}_t\bar{S}'_t)\bar{\Omega}^{-1}_{11.2}\}, \tag{3.46}$$

where

$$\check{\Omega}_{11.2} = \sum_{j=-l}^{l}\check{D}(j)k(j/l), \; \check{D}(j) = T^{-1}\sum_{t=1}^{T-j}\Delta\check{S}_t\Delta\check{S}'_{t+j}, \tag{3.47}$$

and

$$\bar{\Omega}_{11.2} = \sum_{j=-l}^{l}\bar{D}(j)k(j/l), \; \bar{D}(j) = T^{-1}\sum_{t=1}^{T-j}(\bar{y}_t - \bar{B}\bar{q}_t)(\bar{y}_{t+j} - \bar{B}\bar{q}_{t+j})'. \tag{3.48}$$

Note that $\tilde{\kappa}$ and $\tilde{\Sigma}$ are consistent estimates of κ and Σ, respectively, which use $\{\tilde{w}_t\}$ and $\{\Delta x_t\}$. Regarding the lag truncation number l for $\Omega^*_{11.2}$, we also assume A8 and A9. Note that $\check{\Omega}_{11.2}$ and $\bar{\Omega}_{11.2}$ are consistent estimates of $\Omega_{11.2} = \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21}$.

The asymptotic null distributions of the tests we have introduced are reported in the following theorem.

Theorem 1. *Suppose that assumptions A1-A9 in Section 2 hold true. Under the null hypothesis* (3.7), *we have:*

(i) For the system of equations (3.1),

$$\begin{aligned} LM_I &\Rightarrow tr[\{\int_0^1 dV^*(r)V^*(r)'\}\{\int_0^1 V^*(r)dV^*(r)')] && (3.49) \\ LM_{II} &\Rightarrow tr[\{\int_0^1 dV^*(r)V^*(r)'\}\{\int_0^1 V^*(r)V^*(r)'dr\}^{-1} \\ &\quad \cdot\{\int_0^1 V^*(r)dV^*(r)'\}] && (3.50) \\ SBDH &\Rightarrow tr[\int_0^1 V^*(r)V^*(r)'dr] && (3.51) \end{aligned}$$

where

$$\begin{aligned} V^*(r) &= W_1(r) - \{\int_0^1 dW_1(s)W_2(s)'\}\{\int_0^1 W_2(s)W_2(s)'ds\}^{-1}\bar{W}_2(r), && (3.52) \\ dV^*(r) &= dW_1(r) - \{\int_0^1 dW_1(s)W_2(s)'\} \\ &\quad \cdot\{\int_0^1 W_2(s)W_2(s)'dr\}^{-1}W_2(r)'dr, && (3.53) \end{aligned}$$

$W_1(r)$ *and* $W_2(r)$ *are independent standard vector Brownian motion of size* n *and* m, *respectively, and* $\bar{W}_2(r) = \int_0^r W_2(s)ds$.

(ii) For the system of equations (3.2)

$$LM_I \Rightarrow tr[\{\int_0^1 d\check{V}(r)\check{V}(r)'\}\{\int_0^1 \check{V}(r)d\check{V}(r)'\}] \qquad (3.54)$$

$$\begin{aligned} LM_{II} &\Rightarrow tr[\{\int_0^1 d\check{V}(r)\check{V}(r)'\}\{\int_0^1 \check{V}(r)\check{V}'(r)dr\}^{-1}\{\int_0^1 \check{V}(r)d\check{V}(r)'\}] && (3.55) \\ SBDH_I &\Rightarrow tr[\int_0^1 \check{V}(r)\check{V}(r)'dr] && (3.56) \\ SBDH_{II} &\Rightarrow tr[\int_0^1 \bar{V}(r)\bar{V}(r)'dr] && (3.57) \end{aligned}$$

where

$$\begin{aligned} \check{V}(r) &= W_1(r) - \{\int_0^1 W_1(s)S_w(s)'ds\}\{\int_0^1 S_w(s)S_w(s)'ds\}^{-1}S_w(r), && (3.58) \\ d\check{V}(r) &= dW_1(r) - \{\int_0^1 W_1(s)S_w(s)'ds\} \\ &\quad \{\int_0^1 S_w(s)S_w(s)'dr\}^{-1}Q_w(r)'dr, && (3.59) \\ \bar{V}(r) &= W_1(r) - \{\int_0^1 dW_1(s)Q_w(s)'ds\}\{\int_0^1 Q_w(s)Q_w(s)'ds\}^{-1}S_w(r), && (3.60) \\ Q_w(r) &= [R(r)', W_2(r)']', && (3.61) \\ R(r) &= [1, r, \cdots, r^p]', && (3.62) \\ S_w(r) &= \int_0^r Q_w(s)ds. && (3.63) \end{aligned}$$

Remarks:

(*a*) This theorem shows that the tests based on CCR residuals are free of nuisance parameters in the limit. As will be shown in Section 5, this property is not shared by OLS residuals; the asymptotic distributions of the tests based on the OLS residuals involve nuisance parameters which are difficult to eliminate.

(*b*) Now we explain how the asymptotic results in Part (ii) of the above theorem also apply to the case where the regressor x_t contains time trends as in equation (3.4) and Δx_t^0 is estimated by an auxiliary OLS regression. The OLS regression residuals from equation (3.2) are numerically invariant to the presence of time trends in the regressor x_t by a standard theory in linear regression. Further, $\Delta\hat{x}_t^0 \xrightarrow{p} \Delta x_t^0$ for all t. Therefore, the probability limits of the moment estimates required for CCR transformations are asymptotically invariant to the presence of time trends in x_t. The transformed true residual in the presence of time trends in x_t is written as $\bar{u}_t = u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}\Delta\hat{x}_t^0 - (\bar{A} - A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{w}_t$, where $\tilde{w}_t = (\tilde{u}_t', \Delta\hat{x}_t^{0\prime})'$ and the estimates $\tilde{\Omega}$, $\tilde{\Lambda}$ and $\tilde{\Gamma}$ are based on $\tilde{w}_t$. Because the OLS estimate $\bar{A}$ are invariant to the presence of time trends, the CCR residuals obtained by projecting $\{\bar{u}_t\}$ onto the orthogonal complement of the space spanned by $\{c_t, \bar{x}_t\}$ are numerically equivalent to the CCR residuals obtained by projecting $\ddot{u}_t = u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}\Delta x_t^0 - (\bar{A} - A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{w}_t$ onto the same space. This implies that the asymptotic results we have obtained under the assumption that x_t does not contain time trends also hold true when x_t contains time trends as in equation (3.4). But note that the tests statistics using $\Delta\hat{x}_t^0$ and those using Δx_t^0 are not numerically equivalent because the moment estimates for the CCR transformations are different in the two cases, though they converge in probability to the same limits and, therefore, do not affect asymptotic distributions of the tests.

(c) When $n = 1$ (the case of a single equation), the asymptotic distributions of $SBDH$ and $SBDH_{II}$ reduce to those in Shin (1993). In this sense, the tests generated within our framework include Shin's (1993) tests as special cases.

(d) It appears difficult to obtain analytic forms of the pdf's and cdf's of the limiting distributions in this theorem. Hence, we tabulated the percentiles of these distribution by using simulation. The number of iterations was set at 50,000, and normal numbers were generated by using the GAUSS procedure RNDN. The percentiles are reported in Table 5-15 for model (3.1) and for model (3.2) up to $p = 1$.

The tests proposed in this section are consistent as shown in the next theorem.

Theorem 2. *Under the alternative hypothesis* (3.8), *we have:*

(i) For the system of equations (3.1),

$$LM_I = O_p(T^{2(1-\delta)}), LM_{II} = O_p(T^{1-\delta}), SBDH = O_p(T^{1-\delta}). \tag{3.64}$$

(ii) For the system of equations (3.2),

$$LM_I = O_p(T^{2(1-\delta)}), LM_{II} = O_p(T^{1-\delta}), SBDH_I = O_p(T^{1-\delta}), SBDH_{II} = O_p(T^{1-\delta}), \tag{3.65}$$

where $0 < \delta < \frac{1}{2}$.

Remarks:

(a) In light of these results, we reject the null when the computed values of the tests are greater than the corresponding critical values.

(b) These results show that the rate of divergence depends on the divergence rate of lag truncation number. It is expected that the finite sample power of the tests is higher when the lag truncation number grows slower. However, such automatic lag selection methods as Andrews (1991) and Andrews and Monahan (1992) let the lag estimator ($\hat{l}$) grow at the rate of $O_p(T)$, which make the tests inconsistent (see Choi (1992b) for detailed discussions on this issue). One way of avoiding this problem is to use the automatic lag selection methods with a restriction. That is, we estimate the lag truncation number by one of the automatic selection methods, but we let $\hat{l} = c$ (constant) if $\hat{l} > T^{\epsilon}$ where $\frac{1}{2} < \epsilon < 1$. Because $\hat{l} = O_p(T^{\delta})$ $(0 < \delta < \frac{1}{2})$ under the null, this restriction does not affect the lag length estimation under the null at least asymptotically. However, asymptotically, the lag length will be chosen a finite constant under the alternative because $\hat{l} = O_p(T)$. This implies that the tests diverge at faster rates under the alternative (*i.e.*, $\delta = 0$) with this restriction. The tests using this restricted lag selection method perform well in finite samples as will be shown in Section 6.

(c) Using the same arguments as in Remark (ii) following Theorem 1, it is straightforward to show that Part (ii) of this theorem holds true for the case where the regressor x_t contains time trends and Δx_t^0 is estimated by an auxiliary OLS regression.

3.5 Tests Based on Other Estimation Methods

The tests we have proposed are formulated by using CCR residuals. In this section, however, we will show that other efficient estimation methods also yield tests for the null of cointegration which are free of nuisance parameters in the limit. Using the dynamic OLS methods (cf. Stock and Watson (1993) and Saikkonen (1991)) to formulate tests for the null of cointegration in a single equation is studied in Shin (1993). Therefore, we will consider only Phillips and Hansen's FM-OLS methods which are similar to the CCR methods in the sense that preliminary OLS results are used to obtain efficient estimates. We will focus on model (3.1) only in this section, because extending our discussions to model (3.2) is a straightforward exercise.

The FM-OLS estimator for model (3.1) is defined as follows:

$$\dot{A}_{FM} = (\sum_{t=1}^{T} y_t^{+} x_t' - T\hat{\kappa}\hat{\Gamma}_2)(\sum_{t=1}^{T} x_t x_t')^{-1}, \tag{3.66}$$

where $y_t^{+} = y_t - \hat{\Omega}_{12}\hat{\Omega}_{22}^{-1}\Delta x_t$. The asymptotic distribution of the FM-OLS estimator is given in Phillips and Hansen (1990) as

$$T(\dot{A}_{FM} - A) \Rightarrow \{\int_0^1 dB_{1.2}(r)B_2(r)'\}\{\int_0^1 B_2(r)B_2(r)'dr\}^{-1}, \tag{3.67}$$

which is the same as the asymptotic distribution of the CCR estimator. Now we may use the residuals

$$\dot{u}_t^+ = y_t^+ - \dot{A}_{FM}x_t = u_t - \hat{\Omega}_{12}\hat{\Omega}_{22}^{-1}\Delta x_t - (\dot{A}_{FM} - A)x_t \tag{3.68}$$

in order to formulate the *LM* and *SBDH* tests. Because $T(\dot{A}_{FM} - A)$ and $T(A^* - A)$ have the same asymptotic distribution, it is easy to verify that the *LM* and *SBDH* tests based on FM-OLS residuals share the same limiting distributions with those based on CCR residuals. Therefore, the percentiles in Table 3.1-3.3 can also be used for the cointegration tests based on FM-OLS residuals. This analysis shows that it is possible to formulate the nuisance-parameter-free tests for the null of cointegration by using FM-OLS residuals and provides an answer to the problem raised in Tanaka (1993, p. 54).

However, unlike the case of testing the null of non-cointegration (cf. Engle and Granger (1987), Phillips and Ouliaris (1988, 1990), Choi (1992d)), it seems difficult to eliminate the nuisance parameters in the limit if we use OLS residuals for the *LM* and *SBDH* tests. We will illustrate this for model (3.1). The OLS residuals from equation (3.1) can be written as

$$\hat{u}_t = y_t - \hat{A}x_t = u_t - (\hat{A} - A)x_t \tag{3.69}$$

and its partial sum as

$$\hat{S}_t = S_t^u - (\hat{A} - A)S_t^x, \tag{3.70}$$

where $S_t^u = \sum_{i=1}^t u_i$ and $S_t^x = \sum_{i=1}^t x_i$. Therefore, it follows that

$$\begin{aligned} T^{-1}\sum_{t=2}^{T} \hat{S}_{t-1}\Delta\hat{S}_t' &= T^{-1}\sum_{t=2}^{T}\{S_{t-1}^u - (\hat{A}-A)S_{t-1}^x\}\{u_t - (\hat{A}-A)x_t\} \\ &\Rightarrow \int_0^1 \{B_1(r) - \alpha\bar{B}_2(r)\}\{dB_1(r) - \alpha B_2(r)\}' + \Sigma_{11} \quad (3.71) \end{aligned}$$

and that

$$\begin{aligned} T^{-2}\sum_{t=1}^{T} \hat{S}_t\hat{S}_t' &= T^{-2}\sum_{t=1}^{T}\{S_t^u - (\hat{A}-A)S_t^x\}\{S_t^u - (\hat{A}-A)S_t^x\}' \\ &\Rightarrow \int_0^1 \{B_1(r) - \alpha\bar{B}_2(r)\}\{B_1(r) - \alpha\bar{B}_2(r)\}'dr, \quad (3.72) \end{aligned}$$

where $\alpha = \{\int_0^1 B_2(r)dB_1(r)' + \Gamma_{21}\}\{\int_0^1 B_2(r)B_2(r)'\}^{-1}$. These results show that eliminating nuisance parameters is not easy unless x_t is strictly exogenous. We can also make the same argument for model (3.2) by extending these results.

3.6 Finite Sample Power

In this section, we investigate the finite sample performance of the tests we have studied by using simulation. In particular, we will compare the testing strategy of applying univariate cointegration tests several times to each equation of the possibly cointegrated system with that of applying the multivariate tests once to the system of equations.

Unknown parameters in designing experiments are H, A, x_0, u_0, $\{C_i\}$, Ψ, n, m, sample size T and the lag truncation number l. Also, the tests depend on the spectral window $k(\cdot)$, for which we chose the quadratic spectral window following suggestions in Andrews (1991). By a standard theory in linear regression, the tests are invariant to H and A. When a constant term is included in regressions, the tests are also invariant to x_0. However, the tests depend on x_0, when the constant term is excluded. The initial variable u_0 affects the tests under the null. When every element of u_t is $I(1)$ under the alternative, the tests are invariant to u_0. But when at least one element of u_t is $I(0)$, u_0 affects the tests. We will set $u_0 = 0$ and $x_0 = 0$ in the following experiments. Further, partitioning $\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}$, we find by a standard theory in linear regression that the residuals are invariant to Ψ_{11} and so are the tests. Additionally, Ψ_{22} does not affect the values of the tests, because the tests are invariant to nonsingular linear transformations. But the tests depend on Ψ_{12}. We set $\Psi = \begin{bmatrix} 1.5 & 0.3 & 0.2 & 0.1 \\ 0.3 & 1.0 & 0.4 & 0.1 \\ 0.2 & 0.4 & 2.0 & 0.1 \\ 0.1 & 0.1 & 0.1 & 1.1 \end{bmatrix}$, so that x_t and u_t are contemporaneously correlated. The lag truncation number will be chosen automatically by using Andrews' (1991) methods with VAR(4) approximations. But we put a restriction that $\hat{l} = 2$ if $\hat{l} \geq T^{\epsilon}$. This restriction makes the tests consistent as explained in Remark (ii) following Theorem 2. We chose $\epsilon = 0.70$ for model (3.1), $\epsilon = 0.65$ for model (3.2) with an intercept and $\epsilon = 0.60$ for model (3.2) with an intercept and a linear time trend. For the following simulation results, we changed parameters $\{C_i\}$ and the sample size T for $n = m = 2$ (two equations with two regressors) in order to study the finite sample size and power of the tests.

Empirical power and size were calculated out of 5,000 iterations at $T = 100$, 200, 400 by using the critical values reported in Table 5-15. We set the significance level at 0.05 for univariate tests. When the univariate tests are used, the null hypothesis is rejected when the null of cointegration is rejected for either of the two equations.

Therefore, the nominal frequency of rejection for the univariate tests applied to the two equations is $1 - 0.95^2 \simeq 0.1$. Accordingly, we set the significance level for the multivariate tests at 0.10 for meaningful comparisons.

In Table 16, we report the empirical size of the univariate and multivariate tests. Data were generated by

$$u_t = \begin{bmatrix} 0.8 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} u_{t-1} + e_{1t}, x_t = x_{t-1} + e_{2t}, e_t = \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} \equiv iid\ N(0, \Psi). \tag{3.73}$$

Note that each component of u_t is $I(0)$, so that the system is cointegrated. Further, each component of u_t is serially and contemporaneously correlated. We report the size of the tests for the cointegrated system without time trends (model (3.1)) in Part (a), for the cointegrated system with an intercept (model (3.2) with $p = 0$) in Part (b) and for the cointegrated system with an intercept term and a linear time trend (model (3.2) with $p = 1$) in Part (c). In Part (a), we find that the multivariate tests tend to show size distortions even at $T = 400$. Specifically, the LM_I and $SBDH$ tests reject too often and the LM_{II} test rejects too infrequently as compared to the nominal size 0.10. The univariate tests also show serious size distortions except the LM_I test. The empirical size of the LM_I tests is close to 0.10 at T = 200, 400. Overall, it is found that the multivariate tests reject more often than their univariate counterparts and the univariate LM_I test outperforms the rest. In Part (b), the multivariate tests are shown to have less size distortion as compared to Part (a) except $SBDH_{II}$. Notably, the empirical sizes of LM_I and $SBDH_I$ are reasonably close to 0.10 at $T =$ 200, 400. But the empirical size of the $SBDH_{II}$ test is greater than 0.20 at each sample size. The univariate LM_I and LM_{II} tests reject too infrequently. As a matter of fact, the empirical size is close to zero for these tests for all the sample sizes we considered. The univariate $SDBH_I$ test keeps the nominal size well, but the $SBDH_{II}$ test shows serious size distortions. Comparing the univariate and multivariate tests, the multivariate LM_I test performs better than its univariate counterpart, but the univariate $SBDH_I$ test outperforms its multivariate counterpart. The univariate and multivariate LM_{II} tests show similar finite sample properties. In Part (c), we have observations similar to those for Part (b). As in Part (b), the univariate $SBDH_I$ and multivariate LM_I tests keep the nominal size relatively well.

In Table 17, we report the empirical power of the univariate and multivariate tests for the data generated by

$$u_t = \begin{bmatrix} 1.0 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} u_{t-1} + e_{1t}, x_t = x_{t-1} + e_{2t}, e_t = \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} \equiv iid\ N(0, \Psi). \qquad (3.74)$$

Because each element of u_t is $I(1)$, the whole system is not cointegrated. In Part (a), the multivariate tests are shown to have higher power than the univariate tests, but this is well expected from the size properties of these tests. In Part (b), the multivariate LM_I test, which has stable size as shown in Table 16, is much more powerful than its univariate counterpart. But in the case of the $SBDH_I$ tests, the univariate tests is more powerful than its multivariate counterpart. In addition, the LM_{II} test is shown to have low power. In Part (c), we have results similar to those in Part (b). Once again, the multivariate LM_I and univariate $SBDH_I$ tests outperform their corresponding counterparts, and LM_{II} test has low power.

In Table 18, we consider the data generating process

$$u_t = \begin{bmatrix} 1.0 & 0.2 \\ 0.0 & 0.8 \end{bmatrix} u_{t-1} + e_{1t}, x_t = x_{t-1} + e_{2t}, e_t = \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} \equiv iid\ N(0, \Psi). \qquad (3.75)$$

where $u_t^{(1)} = I(1)$ and $u_t^{(2)} = I(0)$. As before, each element of u_t is both serially and contemporaneously correlated. In Part (a), we find that the multivariate tests reject more often than their univariate counterparts except $SBDH$ at $T = 200$, 400. Also, the power is almost the same as that for Part (a) in Table 17. This implies that the number of unit roots in u_t does not have much impact on the finite sample power of the tests. In Parts (b) and (c), we find that the power properties are almost similar to those we observed in Table 17. That is, the multivariate LM_I and univariate $SBDH_I$ tests outperform their corresponding counterparts, and LM_{II} test has low power.

To sum up our findings,

(i) The multivariate LM_I test shows more stable size and is more powerful than its univariate counterpart for models with time trends.

(ii) Both the univariate and multivariate LM_{II} tests show low power.

(iii) The univariate $SBDH_I$ test showd more stable size and is more owerful than its multivariate counterpart.

However, it needs to be borne in mind that our simulation results depend on the specific experimental format we chose. Therefore, the findings we have obtained are at best tentative and we may need more experiments with different experimental formats to fully characterize the finite sample performance of the tests we have proposed.

3.7 Summary and Further Remarks

We have proposed various tests for the null of cointegration that can be applied to a system of equations as well as to a single equation. The tests use CCR residuals to eliminate the nuisance parameters in the limit. The asymptotic distributions of these tests were derived and it was shown that the tests are consistent. Further, we considered the tests based on FM-OLS and OLS residuals. It was shown that we may use FM-OLS residuals instead of CCR residuals without bringing any changes to the asymptotic distributions of the tests. But using OLS residuals was shown to cause difficulties. Simulation was performed to evaluate the finite sample performance of the tests. The multivariate LM_I and univariate $SBDH_I$ tests were shown to work reasonably well in finite samples according to our experimental format.

CHAPTER IV

Testing the Null of Stationarity with Structural Break for Multiple Time Series

4.1 Introduction

Since the analysis of Nelson and Plosser (1982), a great deal of research has been devoted to the unit root hypothesis. Most conventional approaches specify the null to be nonstationary against the alternative of stationarity. However, as suggested by Kwiatkowski, Phillips, Schmidt and Shin (1992, hereafter KPSS), a unit root test should at least be accompanied by stationarity tests for confirmatory data analysis. According to KPSS (1992), many series that have been claimed originally to be $I(1)$ appear to be stationary or inconclusive under stationarity testing. A few test procedures are available for testing the null of stationarity against the alternative of nonstationarity; Park and Choi (1988), Park (1990), Bierens (1991), Herce (1991), Dejong, Nankervis, Savin and Whiteman (1992), Saikkonen and Luukkonen (1989), KPSS (1992), Tanaka (1990), Khan and Ogaki (1992), Stock (1992) and Choi (1992). Choi and Yu (1993) provide a general framework in which many of the tests for $I(k)$ against $I(m+k)$ are generated, and Chapter II of this dissertation developed tests for the null of stationarity for multiple time series.

Another challenging approach against the integrated hypothesis is the structural break hypothesis. Perron (1989) raises this possibility, and suggests that the null hypothesis of unit root be tested against the alternative of stationarity around broken trend. His findings suggest that most of the economic time series appear to be stationary when there is one time crash and the null of a unit-root is rejected for many of the series. Recently, however, Perron (1989) has been criticized for assuming that the structural break points are known, and recent researches by Banerjee, Lumsdaine and Stock (1992), Perron and Vogelsang (1992), Christiano (1992), Zivot and Andrew (1992), to name a few, replace the exogenous breaks with endogenous breaks. Christiano (1992) used the bootstrap method to search for a possible break point in U.S. GNP series and tested whether structural breaks result in spurious behavior of time series. His findings are different from those of Perron (1989). Zivot and Andrew (1992) allow for an unknown structural break and test the unit-root hypothesis against stationarity. They find that there is less evidence against the unit-root hypothesis than in Perron (1989). Amsler and Lee (1994) extends unit-root test suggested in Schimidt and Phillips (1992) to test the null of unit root against the alternative of stationarity with structural change.

So far, most tests for structural break are designed to distinguish the null of nonstationarity against the alternative of stationarity around the mean or trend with structural breaks. There is, however, no procedure available for testing the null of stationarity with a structural break against the alternative of nonstationarity in univariate as well as multiple time series. This is because it is impossible to test the null of structural break against the alternative of parameter constancy. OLS estimators under the structural breaks are consistent even when there is no structural break. Hence, test statistics based on the model with a structural break fail to diverge under the alternative as $T \rightarrow \infty$. In this chapter, we suggest test statistics allowing a structural break under the null of stationarity which diverge under the alternative of nonstationarity.

We can derive some important benefits by testing the null of stationarity with a structural break against nonstationarity. First, we can use the tests for confirmatory data analysis and avoid possible misinterpretation of conventional unit test results. When both tests result in the same conclusion, we can infer the statistical properties with greater confidence. If the tests disagree, we may conclude that the data is not informative along the line of KPSS (1992). Second, stationarity tests avoid the point null hypothesis so that rejecting the null hypothesis can be thought of as evidence in favor of nonstationarity. Thirdly, we may be able to distinguish a stationary series with broken trend from a nonstationary series.

The purpose of this chapter is to introduce tests for the null of stationarity with multiple structural breaks at possibly unknown break points. The tests are designed to handle univariate series as well as multivariate time series. To allow for unknown break points, we take the supremum of the test statistics along the line of Zivot and Andrew (1992). Our test statistics are variants of the tests for the null of stationarity and the null of cointegration suggested in Chapter II and III. All limiting distributions are represented by the product of a multivariate Brownian bridge with structural break parameter, λ. We also report simulation results that study the finite sample performance of the tests. In addition, we will compare the strategy of applying the univariate tests many times and that of using the multivariate tests in finite samples.

This chapter is organized as follows. Section 2 introduces the model and hypotheses. Section 3 considers the effect of structural break on stationarity tests studied in Chapter II. Section 4 derives the limiting distribution of our tests for general time series with known structural break points. Section 5 considers examples. Section 6 extends the tests in Section 4 to the case of unknown break points. Section 7 reports simulation results. Section 8 concludes with a summary and further remarks. All proofs are in the Appendix C.

A few words of notation: All the limits are taken as "$T \to \infty$" unless otherwise specified. Weak convergence is denoted as " $\Rightarrow$". Additionally, "Δ" signifies the usual difference operator. The standard n-vector Brownian motion is written as "$W(r)$" and "$f_{vv}(\cdot)$" denotes the spectral density matrix for $\{v_t\}$. The indicator function is represented by "ι". Lastly, "$A^{(i,j)}$" denotes the $(i,j) - th$ elements of the matrix A.

4.2 The Models, Hypotheses and Assumptions

We consider the system of equations

$$y_t = Ac_t + x_t, \tag{4.1}$$

where y_t represents an $n \times 1$ vector time series, c_t represents a $(p+1) \times 1$ vector of time polynomials and A represents an $n \times (p+1)$ parameter matrix, respectively. Specifically, $c_t = [1, t, \cdots, t^p]'$, with a suitable weight matrix δ_T, $\delta_T^{-1} c_{[Tr]} \to c(r)$ in $D[0,1]$. Obviously, $\int_0^1 cc'$ is nonsingular and positive definite (see Park (1990,1992)).

In this case, $\delta_T = diag[1, T, \cdots, T^p]$ and $c = [1, r, \cdots, r^p]'$. Also, we can transform equation (4.1) as in Chapter II and III and in Choi and Yu (1993). After summing up equation (4.1), we have following equation:

$$P_t = Ag_t + S_t, \tag{4.2}$$

where $P_t = \sum_{j=1}^{t} y_j$, $g_t = \sum_{j=1}^{t} c_j$, and $S_t = \sum_{j=1}^{t} x_j$.

Our main interest is in testing whether the time series x_t is stationary when there exist multiple structural breaks. Specifically, we are interested in testing the null hypothesis

$$H_0 : x_t = I(0) \text{ with structural breaks.} \tag{4.3}$$

against the alternative.

$$H_1 : x_t^{(i)} = I(k_i),\ k_i \geq 1 \text{ for some } i. \tag{4.4}$$

The null hypothesis (4.3) is equivalent to that every series in the system of equations given by equation (4.1) is stationary, possibly around time trend of proper order with structural breaks. Under the alternative, we allow each element of x_t to have a different order of integration but require that at least one element be nonstationary.

Letting $w_t = x_t$, we assume under the null that w_t satisfies the assumptions A1-A9 in chapter II.

A stationary and invertible vector ARMA process is a special case of $\{w_t\}$. Under A1, A2, A4, A5 and A6, we have, as in Phillips and Solo (1992, p. 985),

$$T^{-1/2} \sum_{t=1}^{[Tr]} w_t \Rightarrow B(r), \tag{4.5}$$

where $B(r)$ is a Brownian motion with covariance matrix Ω and $[x]$ denotes the integer part of x. Also, extending Hannan and Heyde's (1972) results, we have under A1, A4, A5 and an assumption implied by A2 that

$$T^{-1}\sum_{t=1}^{T} w_t w_t' \xrightarrow{p} \Sigma, \tag{4.6}$$

where $\Sigma = E(w_t w_t') = \sum_{i=0}^{\infty} C_i \Psi C_i'$. A3 is required to ensure that the limiting distribution of the partial sum process in (4.5) is non-degenerate and to ensure that $\{w_t\}$ does not have an MA unit root. A7 implies that $\sum_{i=1}^{t} w_i$ is not cointegrated under the null hypothesis.

4.3 Test Statistics and the Effect of Structural Breaks

By defining an appropriate set of parameters A and regressors c_t of equation (4.1), the null hypothesis under structural breaks could be formulated. The test statistics we are going to consider are those studied in Chapter II and III, which are given below:

$$LM_I = tr\{(T^{-1}\sum_{t=2}^{T} \Delta\tilde{S}_t\tilde{S}_{t-1}' - \tilde{\Omega}_1')\tilde{\Omega}_l^{-1}(T^{-1}\sum_{t=2}^{T} \tilde{S}_{t-1}\Delta\tilde{S}_t' - \tilde{\Omega}_1)\tilde{\Omega}_l^{-1}\}, \tag{4.7}$$

$$LM_{II} = tr\{(\sum_{t=2}^{T} \Delta\tilde{S}_t\tilde{S}_{t-1}' - T\tilde{\Omega}_1')(\sum_{t=2}^{T} \tilde{S}_{t-1}\tilde{S}_{t-1}')^{-1}(\sum_{t=2}^{T} \tilde{S}_{t-1}\Delta\tilde{S}_t' - T\tilde{\Omega}_1)\tilde{\Omega}_l^{-1}\}, \tag{4.8}$$

$$SBDH_I = tr\{(T^{-2}\sum_{t=1}^{T} \tilde{S}_t\tilde{S}_t')\tilde{\Omega}_l^{-1}\}, \tag{4.9}$$

and

$$SBDH_{II} = tr\{(T^{-2}\sum_{t=1}^{T} \bar{S}_t\bar{S}_t')\bar{\Omega}_l^{-1}\}, \tag{4.10}$$

where the bar (−) denotes residuals obtained from equation (4.1) and the tilde (∼) from equation (4.2). To understand the impact of a structural break on the stationarity tests, consider the null of stationarity against nonstationarity studied in Chapter II. Suppose there is a one time pure structural break at $T_B = T\lambda$ for $\lambda \in (0,1)$ following Andrew (1992). Then, the equation (4.1) should be modified as below.

$$\begin{aligned} y_t &= A_1 c_t + x_t,\ t = 1, \cdots, T\lambda, && (4.11) \\ y_t &= A_2 c_t + x_t,\ t = T\lambda + 1, \cdots, T, && (4.12) \end{aligned}$$

where $\lambda \in (0,1)$ denotes the break point, and $A_1 \neq A_2$. Let $\iota_1 = 1$ if $t \leq T\lambda$ or 0 otherwise and $\iota_2 = 1$ if $t > T\lambda$ or 0 otherwise. The vector of indicator functions is denoted by $\iota = [\iota_1,\ \iota_2]'$. Equation (4.12) can then be written using the indicator functions as below.

$$\begin{aligned} y_t &= A_1 c_t \iota_1 + A_2 c_t \iota_2 + x_t, \\ &= A d_t + x_t, && (4.13) \end{aligned}$$

where $d_t = [\iota_1 c_t',\ \iota_2 c_t']' = \iota \otimes c_t$ and $A = [A_1\ A_2]$. This specification is very simple but useful for our purposes. Further, it is easy to formulate various structural breaks by defining the parameter matrix A and the regressors d_t given the structural break point. Test statistics suggested in this chapter are obtained from equation (4.13) though there are possible alternative expressions. Clearly, (4.1) is misspecified under the null hypothesis of stationarity with structural breaks, and we expect that the omitted deterministic component would be big enough to cause the estimated residuals nonstationary hence cause the test statistics diverge. The following theorem states the effect of a one time structural break on the tests for stationarity.

Theorem 1. *Suppose that assumptions A1 - A9 hold and that the time polynomial of order p in the regression equation is correctly specified. Then under the null hypothesis with a one time structural break,*

$$(i)\, LM_I = O_p(T^{2(1-\delta)}), \tag{4.14}$$

$$(ii)\, LM_{II} = O_p(T^{1-\delta}), \tag{4.15}$$

$$(iii) SBDH_I = O_p(T^{1-\delta}), \tag{4.16}$$

$$(iv) SBDH_{II} = O_p(T^{1-\delta}), \tag{4.17}$$

where $0 < \delta < \frac{1}{2}$.

Remarks:

(a) These results indicate that if there exists a structural break we always reject the null of stationarity asymptotically even when x_t is $I(0)$. Therefore, rejecting the null of stationarity does not automatically imply acceptance of the alternative of nonstationarity; it could be an indication of structural break instead.

(b) The results are consistent with Perron (1989) in the sense that a one time structural break could make stationary time series behave as if it were a nonstationary series.

(c) The rates of divergence of the test statistics are the same as those under the alternative of nonstationarity.

(d) These divergent results are also expected in case of other stationarity tests. The effects are exactly the opposite when conventional unit root tests such as ADF and Z_α are considered. That is, these unit root tests become inconsistent when a series contains a broken trend.

4.4 Model with Known Structural Break Points

In this section, we will demonstrate how we can effectively allow for various structural breaks and focus on the stationarity of the series we want to assess.

4.4.1 Model with a one time structural break

Suppose there is one time structural break at the point $T_B = T\lambda$. The model is given by equation (4.13)

$$y_t = Ad_t + x_t. \tag{4.18}$$

In what follows, we will consider three types of structural breaks: a pure structural break, a partial structural break and a continuous broken trend. All of these types of structural breaks could be allowed in equation (4.13).

Case 1: Pure structural break

A pure structural break is defined as the case in which all coefficients of the equation change their value at $T\lambda$. The parameter matrix A is $n \times k$. Then, we set $d_t = (\iota \otimes c_t)$ with dimension $k = 2 \times (p+1)$. Obviously, $(I_2 \otimes \delta_T^{-1})d_{[Tr]} \to f = (\iota \otimes c)$. The limiting distribution of the OLS estimator is given by

$$T^{1/2}(\bar{A} - A)(I_2 \otimes \delta_T) \Rightarrow \int_0^1 dBf'(\int_0^1 ff')^{-1} = N(0, \Omega \otimes (\int_0^1 ff')^{-1}). \tag{4.19}$$

Note that $\int_0^1 ff' = \int_0^1 (\iota\iota' \otimes cc') = diag[\int_0^1 cc'\iota_1, \int_0^1 cc'\iota_2] = diag[\int_0^\lambda cc', \int_\lambda^1 cc']$, which is nonsingular.

Case 2: Partial structural break

A partial structural break is defined as the case in which some of the coefficients of the equation change their values at $T\lambda$. The $n \times k$ parameter matrix $A = [A_1A_2A_3]$. Rearrange $c_t = [c_{1t},\, c_{2t}]'$ such that the coefficients of $c_{2t}(m \times 1)$ change their values but those of c_{1t} do not change their values. Then, we set $d_t = [c_{1t}',\, c_{2t}'\iota_1,\, c_{2t}'\iota_2]'$ with dimension $k = p + 1 + m$. Letting δ_{1T} and δ_{2T} be appropriate weight matrices such that $\delta_{1T}^{-1}c_{1[Tr]} \rightarrow c_1(r)$, $\delta_{2T}^{-1}c_{2[Tr]} \rightarrow c_2(r)$ so that $diag[\delta_{1T}^{-1}, \delta_{2T}^{-1}, \delta_{2T}^{-1}]d_{[Tr]} \rightarrow f(r) = (c_1', c_2'\iota_1, c_2'\iota_2)'$. The limiting distribution of the OLS estimator is given by equation (4.19) with $\int_0^1 ff' = \int_0^1 \begin{bmatrix} c_1c_1' & c_1c_2'\iota_1 & c_1c_2'\iota_2 \\ c_2c_1'\iota_1 & c_1c_2'\iota_1 & 0 \\ c_2c_1'\iota_2 & 0 & c_2c_2'\iota_2 \end{bmatrix}$.

Case 3: Structural break with continuous restriction

To formulate continuity with a structural break, there must be at least two coefficients that change their values. With one time structural break, the continuity restriction reduces the number of parameters to be estimated by one as compared to case 1 and 2. We can easily modify the equation to guarantee continuity. Without loss of generality, assume a partial structural break at λ for c_{2t}. Then the restriction becomes $A_1c_{1T_B} + A_2c_{2T_B} = A_1c_{1T_B} + A_3c_{2T_B}$ which implies that $A_2c_{2T_B} - A_3c_{2T_B} = 0$. Since $A_2 \neq A_3$, we can solve the restriction for one coefficient. We solve the restriction for the first column of A_2 to obtain

$$A_2^{(1)}c_{2T_B}^{(1)} = -\underline{A}_2\underline{c}_{2T_B} + A_3c_{2T_B}, \tag{4.20}$$

where $\underline{A}_2$ is the $n \times (m-1)$ matrix created by deleting $A_2^{(1)}$, the first column of A_2, from A_2 and $\underline{c}_{2t}$ is $(m-1) \times 1$ vector of constants created by deleting $c_{2t}^{(1)}$, the first element of c_{2t}, from c_{2t}. Using the restriction given by equation (4.20), we express equation (4.1) as

$$\begin{aligned} y_t &= A_1 c_{1t} + A_2 c_{2t} + x_t \\ &= A_1 c_{1t} + \underline{A}_2 \underline{c}_{2t} + A_3 c_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)} - \underline{A}_2 \underline{c}_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)} + x_t \\ &= A_1 c_{1t} + \underline{A}_2 (\underline{c}_{2t} - \underline{c}_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)}) + A_3 c_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)} + x_t \; for \; t \leq T\lambda, \\ y_t &= A_1 c_{1t} + A_3 c_{2t} + x_t \; for \; t > T\lambda. \end{aligned} \tag{4.21}$$

The regression equation becomes

$$\begin{aligned} y_t &= A_1 c_{1t} + \underline{A}_2 (\underline{c}_{2t} - \underline{c}_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)}) \iota_1 + A_3 (c_{2t} \iota_2 + c_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)} \iota_1) + x_t \\ &= \underline{A} d_t + x_t \end{aligned} \tag{4.22}$$

where $d_t = [c_{1t}', \; \iota_1 (\underline{c}_{2t} - \underline{c}_{2T_B} c_{2t}^{(1)} / c_{2T_B}^{(1)})', \; \iota_2 c_{2t}' + \iota_1 c_{2T_B}' c_{2t}^{(1)} / c_{2T_B}^{(1)}]'$. Note that $\underline{A}$ is $n \times k$ with $k = p + m$ which is reduced in dimension by 1 compare to case 2. Also, the limiting distribution of the OLS estimator is invariant to the value of A.

In what follows, we will derive the limiting distributions of the test statistics. When there is an intercept in the model, we can not use the estimated residuals for LM_I and LM_{II} since $\bar{S}_T = 0$ and $\sum_{t=2}^T \Delta \bar{S}_t \bar{S}_{t-1} = \frac{1}{2} (\bar{S}_T \bar{S}_T' - \sum_{t=1}^T \Delta \bar{S}_t \Delta \bar{S}_t')$ which is degenerate. In such a case, we formulate the following regression equation by summing up the equation (4.13) over t as in Chapter II and III and in Choi and Yu (1993).

$$S_t^y = A h_t + S_t, \tag{4.23}$$

where $h_t = \sum_{i=1}^t d_t$ and $S_t^y = \sum_{i=1}^t y_i$. Denoting the residuals $\Delta \bar{x}_t$ and $\tilde{S}_t$ from equations (4.13) and (4.23), respectively, the following test statistics will be considered in this chapter.

$$LM_I = tr\{T^{-1}\sum_{t=2}^{T}\Delta\tilde{S}_t\tilde{S}'_{t-1} - \tilde{\Omega}'_1)\tilde{\Omega}_l^{-1}(T^{-1}\sum_{t=2}^{T}\tilde{S}_{t-1}\Delta\tilde{S}'_t - \tilde{\Omega}_1)\tilde{\Omega}_l^{-1}\} \tag{4.24}$$

$$LM_{II} = tr\{(\sum_{t=2}^{T}\Delta\tilde{S}_t\tilde{S}'_{t-1} - T\tilde{\Omega}'_1)(\sum_{t=2}^{T}\tilde{S}_{t-1}\tilde{S}'_{t-1})^{-1}(\sum_{t=2}^{T}\tilde{S}_{t-1}\Delta\tilde{S}'_t - T\tilde{\Omega}_1)\tilde{\Omega}_l^{-1}\} \tag{4.25}$$

$$SBDH_I = tr\{(T^{-2}\sum_{t=1}^{T}\tilde{S}_t\tilde{S}'_t)\tilde{\Omega}_l^{-1}\}, \tag{4.26}$$

$$SBDH_{II} = tr\{(T^{-2}\sum_{t=1}^{T}\bar{S}_t\bar{S}'_t)\bar{\Omega}_l^{-1}\}, \tag{4.27}$$

where

$$\begin{aligned}
\bar{\Omega}_l &= \sum_{h=-l}^{l}\bar{C}(h)k(h/l), & (4.28)\\
\bar{C}(h) &= \frac{1}{T}\sum_{t=2}^{T-h}\Delta\bar{S}_t\bar{S}'_{t+h}, & (4.29)\\
\tilde{\Omega}_l &= \sum_{h=-l}^{l}\tilde{C}(h)k(h/l), & (4.30)\\
\tilde{C}(h) &= \frac{1}{T}\sum_{t=2}^{T-h}\Delta\tilde{S}_t\tilde{S}'_{t+h}, & (4.31)
\end{aligned}$$

The limiting distributions for the test statistics are presented in the following theorem.

Theorem 2. *Suppose assumptions A1 - A9 hold. (a) Under the null hypothesis with one time structural break at known point* $T_B = \lambda T, \lambda \in (0,1)$*,*

$$(i)\ LM_I \Rightarrow tr\{\int_0^1 d\tilde{W}(r)\tilde{W}(r)'\int_0^1 \tilde{W}(r)d\tilde{W}(r)'\}, \tag{4.32}$$

$$(ii)\ LM_{II} \Rightarrow tr[\int_0^1 d\tilde{W}(r)\tilde{W}(r)'\{\int_0^1 \tilde{W}(r)\tilde{W}(r)'\}^{-1}\int_0^1 \tilde{W}(r)d\tilde{W}(r)'], \tag{4.33}$$

$$(iii)\ SBDH_I \Rightarrow tr\{\int_0^1 \tilde{W}(r)\tilde{W}(r)'\}, \tag{4.34}$$

$$(iv)\ SBDH_{II} \Rightarrow tr\{\int_0^1 \bar{W}(r)\bar{W}(r)'\}, \tag{4.35}$$

where

$$\tilde{W}(r) \;=\; W(r) - \tilde{\psi}h, \tag{4.36}$$

$$\bar{W}(r) \;=\; W(r) - \bar{\psi}h, \tag{4.37}$$

and $\tilde{\psi}$ *and* $\bar{\psi}$ *minimize in* L^2 *norm,*

$$\int_0^1 \|W(r) - \tilde{\psi}h\|^2, \tag{4.38}$$

$$\int_0^1 \|dW(r) - \bar{\psi}f\|^2, \tag{4.39}$$

and $h(r) = \int_0^r d(s)ds$.

(b) Under the alternative hypothesis,

$$(i) LM_I \;=\; O_p(T^{2(1-\delta)}), \tag{4.40}$$

$$(ii) LM_{II} \;=\; O_p(T^{1-\delta}), \tag{4.41}$$

$$(iii) SBDH_I \;=\; O_p(T^{1-\delta}), \tag{4.42}$$

$$(iv) SBDH_{II} \;=\; O_p(T^{1-\delta}), \tag{4.43}$$

where $0 < \delta < \frac{1}{2}$.

Remarks:

(a) The test statistics will not diverge when x_t is stationary without a structural break. These tests are consistent even without a structural break under the alternative and have power against nonstationarity only. This is because consistently estimated residuals are used to construct the test statistics when there is no structural break.

(*b*) From Theorem 1, it is obvious that the test statistics diverge when there is more than one structural break. However, it is straightforward to extend our formulation to multiple structural breaks using additional indicator functions.

(*c*) When there is no structural break, the above theorem is still valid because the parameter estimates as well as the residuals are consistent. They do not have power against no structural break. Therefore, in connection with (*b*), allowing for more structural breaks is asymptotically safe but it may affect the power performance because of efficiency losses due to the increased number of parameters to be estimated.

(*d*) For the case of no structural break ($d_t = c_t$), the asymptotic distributions and the finite sample performances of these statistics are reported in Chapter II.

4.4.2 Model with multiple structural breaks

Consider the case with multiple structural breaks at $\lambda = (\lambda_1, \cdots, \lambda_q)$, $\lambda_i \in (0,1)$, $i = 1, \cdots, q$, for an n-vector time series y_t. Extending model (4.13) to the general case, we consider the following regression equation. Without loss of generality, assume partial structural breaks. The model is thus given by

$$y_t = A_1 c_{1t} + B_1 c_{2t} \iota_1 + \cdots + B_q c_{2t} \iota_q + B_{q+1} c_{2t} \iota_{q+1} + u_t, \tag{4.44}$$

where $\iota_j = 1$ if $T\lambda_{j-1} < t \leq T\lambda_j$, 0 otherwise. $\iota_1 = 1$ if $t \leq T\lambda_1$ and 0 otherwise. $\iota_{q+1} = 1$ if $T\lambda_q < t \leq T$. Again, letting $d_t = [c_{1t}', \iota_1 c_{2t}', \cdots, \iota_q c_{2t}', \iota_{q+1} c_{2t}']'$, we have the same equation as (4.13). Equation (4.44) also allows q breaks. Hence, the limiting distributions for the OLS estimators is given by the equation (4.19) with the proper replacement of f and h. Note that we can also consider the various cases discussed earlier. Note that the deterministic components satisfy $\delta_T^{-1} d_{[Tr]} \to f(r) = [c_1', \iota_1 c_2', \cdots, \iota_q c_2', \iota_{q+1} c_2']'$ and $\int_0^1 f f' > 0$, we have the following theorem.

Theorem 3. *Suppose that assumptions A1 - A9 hold. Then the results of Theorem 2 hold under multiple structural breaks with proper replacement of f and h.*

4.5 Examples and Feasible Models

In this section, we will consider several models with structural breaks that are feasible in applied econometrics. Suppose that there is a one time structural break which could be either a partial structural break or a pure structural break as in Andrews (1992). Without loss of generality, we assume a partial structural break. We will consider models restricted to be continuous at the time of the structural break and models without such a restriction. The models are:

$M(1)$:

$$y_t = a_0 t^0 + a_1 t^1 + \cdots + a_{k-1} t^{k-1} + a_k^1 t^k + \cdots + a_\ell^1 t^\ell + \cdots + a_p t^p + x_t \text{ for } t \leq T_B, \quad (4.45)$$

$$y_t = a_0 t^0 + a_1 t^1 + \cdots + a_{k-1} t^{k-1} + a_k^2 t^k + \cdots + a_\ell^2 t^\ell + \cdots + a_p t^p + x_t \text{ for } t > T_B. \quad (4.46)$$

$M(2) : M(1) +$ continuous at T_B.

Note that $\ell - k + 1$ parameters for $c_{2t} = [t^k, \cdots, t^\ell]'$ change their values at $T\lambda$. Define indicator functions ι_1 and ι_2 such that $\iota_1(t) = 1$ for $t \leq T_B$ (or $r \leq \lambda$) and $\iota_2(t) = 1$

for $t > T_B$ (or. $r > \lambda$), respectively. Then the time polynomial is given by

$$d_t = [1, \cdots, t^{k-1}, t^k \iota_1, t^k \iota_2, \cdots, t^\ell \iota_1, t^\ell \iota_2, t^{\ell+1}, \cdots, t^p]' \tag{4.47}$$

and

$$\begin{aligned} d_t &= [1, \cdots, t^k, t^k(t\iota_1 + T_B\iota_2), t^k(t - T_B)\iota_2, \cdots, t^k(t^{\ell-k}\iota_1 + T_B^{\ell-k}\iota_2), \\ &\quad t^k(t^{\ell-k} - T_B^{\ell-k})\iota_2, t^{\ell+1}, \cdots, t^p]' \end{aligned} \tag{4.48}$$

for $M(1)$ and $M(2)$, respectively. Hence, under the null, it is possible to interpret equation (4.13) as a stationary time series with a structural break. Clearly, the $\ell - k + p + 2$ or $\ell - k + p + 1$ dimensional vector sequences of deterministic trend variables d_t satisfies $\delta_T^{-1}\ d_{[Tr]} \to f(r)$ in $D[0, 1]$ with a suitable weight matrix. In particular, $f(r)$ is given by $f = [1,\ \mathrm{r}, \cdots,\ r^k\iota_1,\ r^k\iota_2, \cdots,\ r^\ell\iota_1, r^\ell\iota_2,\ t^{\ell+1},\ \cdots,\ r^p]'$ and $[1, r, \cdots, r^k, r^k(r\iota_1 + \lambda\iota_2), r^k(r - \lambda)\iota_2, \cdots, r^k(r^{\ell-k}\iota_1 + \lambda^{\ell-k}\iota_2), r^k(r^{\ell-k} - \lambda^{\ell-k})\iota_2, t^{\ell+1},$ $\cdots, r^p]'$ for $M(1)$ and $M(2)$, respectively. The weight matrices are $diag[1, T, \cdots, T^k,$ $T^k, \cdots, T^\ell, T^\ell, T^{\ell+1}, \cdots, T^p]$ and $diag[1, T, \cdots, T^k, T^{k+1}, T^{k+1}, \cdots, T^\ell, T^\ell, T^{\ell+1},$ $\cdots, T^p]$ for $M(1)$ and $M(2)$, respectively.

Those models studied by Perron (1989) and many others are special cases of $M(1)$ and $M(2)$ with $p = 0$ or 1. In particular, we have the following models:

Model 1. pure level shift ($p = 0$)

$$d_t = [\iota_1(t), \iota_2(t)]', \tag{4.49}$$

$$f(r) = [\iota_1, \iota_2]' \tag{4.50}$$

Model 2. partial level shift ($p = 1$)

$$d_t = [\iota_1(t), \iota_2(t), t]', \tag{4.51}$$

$$f(r) = [\iota_1, \iota_2, r]' \tag{4.52}$$

Model 3. pure level/trend shift under a continuity restriction ($p = 1$)

$$d_t = [1, t - (t - T_B)\iota_2(t), (t - T_B)\iota_2(t)]', \tag{4.53}$$

$$f(r) = [1, r - (r - \lambda)\iota_2, (r - \lambda)\iota_2]' \tag{4.54}$$

Model 4. pure level/trend shift without restriction ($p = 1$)

$$d_t = [\iota_1(t), \iota_2(t), t\iota_1(t), t\iota_2(t)]', \tag{4.55}$$

$$f(r) = [\iota_1, \iota_2, r\iota_1, r\iota_2]' \tag{4.56}$$

The limiting distributions of the test statistics for the null of stationarity with a one time structural break are reported in the following lemma.

Lemma 4. *Suppose that assumptions A1 - A9 hold. The results in Theorem 2 hold with*

$$\begin{aligned} h(r) = {} & [r, \frac{r^2}{2}, \cdots, \frac{r^k}{k}, \frac{1}{k+1}(r^{k+1}\iota_1 + \lambda^{k+1}\iota_2), \frac{1}{k+1}(r^{k+1} - \lambda^{k+1})\iota_2, \cdots, \\ & \frac{1}{\ell+1}(r^{\ell+1}\iota_1 + \lambda^{\ell+1}\iota_2), \frac{1}{\ell+1}(r^{\ell+1} - \lambda^{\ell+1})\iota_2, \cdots, \frac{r^{p+1}}{p+1}]' \end{aligned} \tag{4.57}$$

for $M(1)$,

$$\begin{aligned} h(r) = {} & [r, \frac{r^2}{2}, \cdots, \frac{r^k}{k}, \frac{r^{k+1}}{k+1}, \frac{r^{k+2}}{k+2} - [r^{k+1}(\frac{r}{k+2} - \frac{\lambda}{k+1}) - \lambda^{k+2}(\frac{1}{k+2} \\ & - \frac{1}{k+1})]\iota_2, [r^{k+1}(\frac{r}{k+2} - \frac{\lambda}{k+1}) - \lambda^{k+2}(\frac{1}{k+2} - \frac{1}{k+1})]\iota_2, \\ & \cdots, \frac{r^{\ell+1}}{\ell+1} - [r^{k+1}(\frac{r^{\ell-k}}{\ell+1} - \frac{\lambda^{\ell-k}}{k+1}) - \lambda^{\ell+1}(\frac{1}{\ell+1} - \frac{1}{k+1})]\iota_2, \\ & [r^{k+1}(\frac{r^{\ell-k}}{\ell+1} - \frac{\lambda^{\ell-k}}{k+1}) - \lambda^{\ell+1}(\frac{1}{\ell+1} - \frac{1}{k+1})]\iota_2, \frac{r^{\ell+2}}{\ell+2} \cdots, \frac{r^{p+1}}{p+1}]' \end{aligned} \tag{4.58}$$

for $M(2)$.

Remarks:

(a) These tests are consistent even without a structural break under the alternative and have power against nonstationarity only. This is because consistently estimated residuals are used to construct the test statistics when there are no structural breaks.

(b) Specifically, $h(r)$ is given by

$$[r\iota_1 + \lambda\iota_2, (r-\lambda)\iota_2]' \text{ for Model 1,}$$

$$[r\iota_1 + \lambda\iota_2, (r-\lambda)\iota_2, \frac{1}{2}r^2] \text{ for Model 2,}$$

$$[r, \frac{1}{2}r^2\iota_1 + \frac{1}{2}(r^2 - 2\lambda r + \lambda^2)\iota_2, \frac{1}{2}(r^2 - 2\lambda r + \lambda^2)\iota_2]' \text{ for Model 3,}$$

$$[r\iota_1 + \lambda\iota_2, (r-\lambda)\iota_2, \frac{1}{2}r^2\iota_1 + \frac{1}{2}\lambda^2\iota_2, \frac{1}{2}(r^2 - \lambda^2)\iota_2]' \text{ for Model 4.}$$

(c) From Theorem 1, it is obvious that the test statistics diverge when there is more than one structural break. However, it is straightforward to extend our formulation to multiple structural breaks using additional indicator functions.

(d) The asymptotic critical values are tabulated by simulation in Table 19-26 for $\lambda =$ 0.25, 0.33, 0.41, 0.49, 0.59, 0.63, for $n = 1$ and 2, respectively.

Suppose that there are q structural breaks at $T_i = \lambda_i T$ for $\lambda_i \in (0,1), i = 1, \cdots, q$. Again, partial and pure structural breaks are allowed. Without loss of generality, assume $0 < \lambda_1 < \cdots < \lambda_q < 1$. Then, for $M(1)$ and $M(2)$, d_t and d can be written as follow:

$$\begin{aligned} d_t &= [1, t, \cdots, t^{k-1}, t^k\iota_1, \cdots, t^k\iota_q, \cdots, t^\ell\iota_1, \cdots, t^\ell\iota_{q+1}, \cdots, t^{\ell+1}, t^p]' && (4.59) \\ f(r) &= [1, r, \cdots, r^{k-1}, r^k\iota_1, \cdots, r^k\iota_q, \cdots, r^\ell\iota_1, \cdots, r^\ell\iota_{q+1}, \cdots, r^{\ell+1}, r^p]' && (4.60) \end{aligned}$$

and

$$\begin{aligned} d_t &= [1, t, \cdots, t^{k-1}, t^k, t^k(t\eta_1 - (t - T_1)\eta_2), t^k((t - T_1)\eta_2 - (t - T_2)\eta_3), \\ &\quad \cdots, t^k(t - T_q)\eta_{q+1}, \cdots, t^k(t^{\ell-k}\eta_1 - (t^{\ell-k} - T_1^{\ell-k})\eta_2, \\ &\quad \cdots, t^k(t^{\ell-k} - T_q^{\ell-k})\eta_{q+1}, t^{\ell+1}, \cdots, t^p]' \qquad (4.61) \\ f(r) &= [1, r, \cdots, r^{k-1}, r^k, r^k(r\eta_1 - (r - \lambda_1)\eta_2), r^k((r - \lambda_1)\eta_2 - (r - \lambda_2)\eta_3), \\ &\quad \cdots, r^k(r - \lambda_q)\eta_{q+1}, \cdots, r^k(r^{\ell-k}\eta_1 - (r^{\ell-k} - \lambda_1^{\ell-k})\eta_2, \\ &\quad \cdots, r^k(r^{\ell-k} - \lambda_q^{\ell-k})\eta_{q+1}, r^{\ell+1}, \cdots, r^p]', \qquad (4.62) \end{aligned}$$

where $\eta_i = \sum_{j=i}^{q+1} \iota_j$ and ι_i is an indicator function such that $\iota_i = 1$ if $T\lambda_{i-1} < t \leq T\lambda_i$ for $i = 1, \cdots, q+1$, $\lambda_i \in (0,1)$, $T\lambda_0 = 1$ and $T\lambda_{q+1} = T$.

These specification allow us to apply stationarity tests for multiple time series with a different number of structural breaks for different series when we know the maximum number of structural breaks. This is because the parameters and residuals were consistently estimated when the number of breaks for each series is less than the number specified. Clearly, when $k = 0$, $\ell = p = q = 1$, the model reduces to model 3 or model 4. The asymptotic distributions are obtained straightforwardly, and are reported in the following lemma.

Lemma 5. *Under the same conditions in Theorem 3 with multiple structural breaks, the results of Theorem 3 hold with proper replacement of f and h.*

4.6 Asymptotic Distribution with Unknown Break Points

The main criticisms against the results in section 4 and section 5 is that the structural break points are assumed to be known. To allow for unknown changing points, we will take the supremum over the range of λ in the manner of Zivot and Andrew (1992). Define $\sup Q_{iT}$ by taking the supremum of these test statistics over λ in a sample of size T. Here, Q_{iT} denotes LM_I, LM_{II}, $SBDH_I$, and $SBDH_{II}$, and Q_i its limiting distribution, for $i = 1, 2, 3$, and 4, respectively. The limiting distributions are reported in the following theorem.

Theorem 6. *Suppose assumptions A1 - A9 hold.*

(a) Under the null hypothesis with q structural breaks at unknown points $T_i = \lambda_i T$, $\lambda_i \in (0,1)$, $i = 1, \cdots, q$,

$$supQ_{kT} \Rightarrow \sup_{\lambda \in (0,1)} (Q_k), \text{ for } k = 1,2,3,4. \tag{4.63}$$

(b) Under the alternative hypothesis,

$$supQ_{kT} = O_p(T^{2(1-\delta)}), \ k = 1, \tag{4.64}$$

$$supQ_{kT} = O_p(T^{1-\delta}), \ k = 2,3,4. \tag{4.65}$$

where $0 < \delta < \frac{1}{2}$.

Remarks:

(a) The above asymptotic distribution could be used for a properly demeaned and/or detrended series with structural breaks. When there is no structural break, one can use the results in Chapter II.

(b) The simulated percentiles for model 1 to 4 for $n = 1, \cdots, 5$ with $q = 1$ are reported in Table 27-30.

(c) The results are obtained by taking $\lambda \in (0.15, 0.85)$ with interval 0.02.

4.7 Finite Sample Power

In this section, we investigate the finite sample performance of the tests introduced in Sections 6 by using simulation. In particular, we compare the testing strategy of applying univariate tests several times to each component of multiple time series with that of applying multivariate tests to the series. The finite sample size and power of the tests proposed in Sections 6 depend on the sample size T, the lag length l for long-run variance estimation, the lag window chosen, and the parameters associated with the DGP of $\{x_t\}$ (see Schmidt and Phillips (1992) for related analyses). But the finite sample power and size are invariant to Σ because the tests are invariant to nonsingular transformation. Further, the finite sample size depends on the initial variable x_0. But the finite sample power of the tests is invariant to x_0. In this section, however, we have used only the Quadratic spectral lag window and chose $x_0 = 0$ for all the experimental results. The univariate and multivariate tests are expected to reject too often under the null as the initial variable takes larger values (cf. Choi (1992b)).

Random numbers for the simulation results were generated by the GAUSS subroutine RNDN. Empirical power was calculated out of 2,000 iterations at $T = 100$, 200 and 400 by using the critical values reported in Table 27-30. The lag length is selected by Andrews' (1991) method with AR(4) and VAR(4) approximations for univariate and multivariate series, respectively. In order to make the tests consistent, we impose the restriction that $\hat{l} = 2$ if $\hat{l} \geq T^{\epsilon}$, where $\epsilon = 0.65$.

In Table 31, we report the empirical power of LM_I, LM_{II}, $SBDH_I$ and $SBDH_{II}$

for Model 1 to Model 4. Data were generated as

$$x_t = \begin{bmatrix} 1.0 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, x_0 = 0, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}. \qquad (4.66)$$

Each component of the bivariate time series $\{x_t\}$ is $I(1)$; $\{x_t^{(1)}\}$ and $\{x_t^{(2)}\}$ are serially correlated. Note that the size of all the tests depends on the initial variable x_0 in finite samples. We investigated finite sample properties in two ways. First, null of $I(0)$ for each series at 5% significance level was tested. Second, multivariate tests and double application of univariate tests are compared. M signifies the model considered. The results for the univariate tests in Part (b) were obtained by calculating the fraction of replications for which the null of $I(0)$ is rejected for at least one series at the 5% level. Because the nominal frequency of non-rejection for the bivariate series is $0.95^2 = 0.9025$, the numbers for the univariate tests should be compared to $1 - 0.9025 \simeq 0.1$. When the numbers are greater than 0.1, the univariate tests are thought to reject too often under the null. For meaningful comparisons, we calculated the fraction of replications for which the multivariate tests reject the null at the 10% level. In Part (a), the results for the tests on each series are reported. M indicates the model used to test the null hypothesis. That is, $\{x_t^{(i)}\}$ is assumed to be the time series with a structural break corresponding to Model 1 to Model 4 respectively. Consider univariate tests of Model 1. $SBDH_I$ and $SBDH_{II}$ are the most powerful tests for all cases. LM_I is powerful at $T = 400$. LM_{II} is least powerful for all cases. Comparing univariate and multivariate tests, univariate $SBDH_I$ and $SBDH_{II}$ are slightly more powerful than their multivariate counterparts at $T = 100$ and 200, and are equally powerful at $T = 400$. However, multivariate LM_I and LM_{II} are more powerful than their univariate counterparts. Among the multivariate tests, LM_{II} is least powerful. However, at $T = 400$, power increases significantly.

Across models, our general conclusions are still valid. However, it should be noted that the power decreases as we move from Model 1 to Model 4, although at T =400, the $SBDH$ tests become equally powerful across different models. In Part (*b*), the multivariate tests are less sensitive to the choice of model than are their univariate counterparts.

In Table 32, we report the empirical power of LM_I, LM_{II} and $SBDH$. Data were generated as

$$x_t = \begin{bmatrix} 1.0 & 0.2 \\ 0.0 & 0.8 \end{bmatrix} x_{t-1} + e_t, x_0 = 0, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}. \qquad (4.67)$$

Note that $x_t^{(1)} = I(1)$, $x_t^{(2)} = I(0)$ and that $\{x_t^{(1)}\}$ and $\{x_t^{(2)}\}$ are serially correlated. The finite sample power of all the tests does not depend on the initial variable x_0 . In Part (a), we report the power and size of the univariate tests for various models. For $x_t^{(1)}$, univariate tests are more powerful than those of $x_t^{(1)}$ from the DGP 1. It seems that the additional serial correlation of the error contributes to power performance. For $x_t^{(2)}$, size distortion is observed at $T = 100$ and 200. At $T = 400$, $SBDH_I$ and $SBDH_{II}$ maintain the nominal significance level relatively well. However, LM_I and LM_{II} do not reject the null frequently enough.

In Part (*b*), univariate $SBDH_I$ and $SBDH_{II}$ are slightly more powerful than their multivariate counterparts. Multivariate LM_{II} is equally as powerful as $SBDH_I$ and $SBDH_{II}$ at $T = 200$ and 400. Across models, it is observed that univariate tests for $x_t^{(1)}$ become less powerful as we depart from Model 1. For $x_t^{(2)}$, size distortion increases as we deviate from Model 1. Multivariate tests also become less powerful as we move away from model 1. However, multivariate tests are less sensitive to the choice of models than are their univariate counterparts.

In Table 33, we report the empirical size of LM_I, LM_{II}, $SBDH_I$ and $SBDH_{II}$ for the data generated by

$$x_t = \begin{bmatrix} 0.8 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, x_0 = 0, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}. \qquad (4.68)$$

Note that $x_t^{(1)}$, $x_t^{(2)} = I(0)$ and that $\{x_t^{(1)}\}$ and $\{x_t^{(2)}\}$ are serially correlated. The finite sample power of all the tests does depend on the initial variable x_0 . In Part (a), we report the size of the univariate tests. As in Table 32, the univariate tests suffer size distortions at $T = 100$ and $T = 200$. Both $SBDH_I$ and $SBDH_{II}$ keep their nominal size at $T = 400$. Further, it is observed that the size distortion increases when an MA component is included. Again, LM_I and LM_{II} do not reject the null frequently enough. In Part (*b*), size distortion is observed. The size distortion, however, is smaller in multivariate tests than in their univariate counterparts. When $T = 400$, multivariate tests maintain nominal size reasonably well, and $SBDH_I$ and LM_{II} reject the null slightly less than the 10% level. Across models, size distortion is qualitatively the same. However, it disappear, in large samples.

The univariate tests are shown to reject the null less frequently than their multivariate counterparts except in the case of $SBDH_{II}$. However, both sets of tests show serious size distortions at $T = 100$. At $T = 400$, though, the multivariate tests have empirical size reasonably close to 0.1, except in the case of LM_{II}. Comparing the four tests, LM_I, $SBDH_I$ and $SBDH_{II}$ tend to reject more often than LM_{II} in all cases. The results reported in Part (*b*) – (*d*) for Models 2 - 4 are similar to Part (a). To summarize our findings (*i*) multivariate LM_{II} suffers size distortion in a negative direction. When T = 200, 400, LM_{II} become significantly powerful. Multivariate LM_I keep their nominal size at T = 400 and are also powerful. (*ii*) For all models considered, the multivariate tests maintain their nominal size well relative to their univariate counterparts. (*iii*) For all models, all tests suffer from serious size distortions at sample sizes $T = 100$ and 200.

4.8 Summary and Further Remarks

In this chapter, we have introduced tests for the null of stationarity with structural breaks against the alternative of nonstationarity. These tests are applicable to univariate as well as multiple time series which are not available currently. The asymptotic distributions were obtained in a unified manner by using the standard vector Brownian motion and the test consistency was established. The effects of omitted structural breaks were analyzed. Simulation results indicate that the tests we have introduced work reasonably well in finite samples and that using the multivariate tests is a better testing strategy than applying the univariate tests several times to each component of a multiple time series. Among the multivariate tests we introduced, the LM_I, $SBDH_I$ and $SBDH_{II}$ tests show the best performance and are recommended for empirical work.

CHAPTER V

Conclusion and Summary

This dissertation studies various tests for stationarity in the residuals of time series. They are (1) stationarity tests, (2) cointegration tests and (3) stationarity tests allowing structural break. Unlike exisiting conventional tests, those tests studied in this dissertation use stationarity as the null hypothesis and unit root as an alternative hypothesis. Those tests are derived from multivariate AR(1) specification under the *LM* principle.

Chapter II suggests stationarity tests and investigates the finite sample properties. We propose various test statistics for the null of stationarity against the alternative of nonstationarity. Tests using the null of stationarity are at least useful as a confirmatory data analysis tool. The asymptotic distributions were obtained in a unified manner by using the standard vector Brownian motion and the test consistency was established. The effects of misspecifying the order of time trends were also analyzed.

In summary, simulation indicates that the tests we have introduced work reasonably well in finite samples and that using the multivariate tests is a better testing strategy than applying the univariate tests several times to each component of a multiple time series. Note that in the case of both of the multivariate tests and univariate tests that we discuss, the null hypothesis and the alternative are the same, namely that all the time series are stationary vs. at least one is non-stationary. The simulation results show that:

1. The multivariate tests show more stable size than their univariate counterparts when the lag length is chosen as $l = l_2$.

2. The multivariate tests are overall more powerful than their univariate counterparts. The multivariate LM_I tests show most stable size and are most powerful among the multivariate tests in most cases; therefore, the multivariate LM_I tests are preferred to other multivariate tests.

3. For detrended series, the multivariate tests using the automatic lag keep the nominal size reasonably well at $T = 200$ and $T = 400$ and outperform the univariate counterparts using the automatic lag selection in terms of size. Further, the multivariate tests using the automatic lag selection methods are appreciably more powerful than other kinds of tests.

Chapter III studies cointegration tests that can be applied to a system of equations as well as to a single equation. The tests use cointegration as the null hypothesis and no-cointegration as an alternative hypothesis. Limiting distributions for the test statistics are derived and tabulated. To obtain nuisance parameter free test statistics, the CCR (canonical cointegration regression) is used in cointegration regression. It is also shown that other efficient estimators such as FM-OLS could be used to obtain the same analytical results. Simulation was performed to evaluate the finite sample performance of the tests. The simulation results indicates that:

1. The multivariate LM_I test shows more stable size and is more powerful than its univariate counterpart for models with time trends.
2. Both the univariate and multivariate LM_{II} tests show low power.
3. The univariate $SBDH_I$ test shows more stable size and is more powerful than its multivariate counterpart.

Chapter IV of the dissertation proposes stationarity tests that can be applied to multivariate time series as well as to univariate time series allowing structural breaks. In addition, we allow structural break point to be unknown apriori. In connection with Perron (1989), omitted structural breaks cause stationarity tests to diverge and hence reject the null of stationarity asymptotically. To construct consistent tests under the condition, we use stationarity with structural breaks as the null hypothesis against nonstationarity as the alternative hypothesis. Our simulation results show that:

1. Multivariate LM_{II} suffers size distortion in a negative direction. When $T = 200$, 400, LM_{II} become significantly powerful. Multivariate LM_I keeps its nominal size at $T = 400$ and is also powerful.
2. For all the models considered, the multivariate tests maintain their nominal size well relative to their univariate counterparts.
3. For all the models, all the tests suffer from serious size distortions at sample sizes $T = 100$ and 200.

Table 1 Percentiles for LM_I, LM_{II} and $SBDH$
(a) Standard

n		90%	95%	97.5%	99%
n = 1	LM_I	0.7272	2.0185	4.0479	7.9380
	LM_{II}	2.9772	4.1274	5.2750	6.8669
	$SBDH$	1.1936	1.6579	2.1144	2.7697
n = 2	LM_I	4.7612	7.9451	12.0762	18.9540
	LM_{II}	10.3933	12.2266	13.9625	16.1380
	$SBDH$	2.0784	2.6324	3.1842	3.9445
n = 3	LM_I	10.3225	15.5646	21.7677	31.7510
	LM_{II}	21.5690	23.9975	26.1924	28.9080
	$SBDH$	2.8229	3.4218	4.0341	4.8367
n = 4	LM_I	17.8355	25.1326	33.7058	46.3680
	LM_{II}	36.6424	39.6987	42.4923	46.0010
	$SBDH$	3.5272	4.2076	4.8804	5.7193
n = 5	LM_I	26.8098	36.4614	47.1369	63.8845
	LM_{II}	55.4286	59.1478	62.3957	66.4541
	$SBDH$	4.2459	4.9778	5.6900	6.5813
n = 6	LM_I	37.0236	49.1842	62.4234	81.5779
	LM_{II}	78.1676	82.4698	86.1781	90.9577
	$SBDH$	4.8879	5.6733	6.4210	7.3680

Table 1 (continued)
(b) Demeaned

n		90%	95%	97.5%	99%
n = 1	LM_I	0.2485	0.2496	0.2499	0.2500
	LM_{II}	6.4249	7.9974	9.5459	11.4497
	$SBDH_T$	0.1929	0.2477	0.3046	0.3838
	$SBDH_B$	0.3471	0.4589	0.5798	0.7419
n = 2	LM_I	0.7462	0.9338	1.1588	1.5162
	LM_{II}	15.3359	17.4409	19.4965	22.0910
	$SBDH_T$	0.3384	0.4063	0.4739	0.5648
	$SBDH_B$	0.6061	0.7464	0.8880	1.0736
n = 3	LM_I	1.5238	1.8656	2.2551	2.8098
	LM_{II}	27.8297	30.5677	33.0936	36.0653
	$SBDH_T$	0.4728	0.5491	0.6243	0.7238
	$SBDH_B$	0.8440	0.9933	1.1456	1.3395
n = 4	LM_I	2.5228	3.0290	3.5800	4.4059
	LM_{II}	44.1606	47.4113	50.3741	53.8893
	$SBDH_T$	0.6012	0.6859	0.7637	0.8714
	$SBDH_B$	1.0599	1.2355	1.4078	1.6156
n = 5	LM_I	3.7147	4.4186	5.1507	6.1883
	LM_{II}	64.2989	68.1676	71.6155	75.8429
	$SBDH_T$	0.7246	0.8157	0.9014	1.0085
	$SBDH_B$	1.2774	1.4636	1.6402	1.8659
n = 6	LM_I	5.0410	5.9058	6.7655	8.0360
	LM_{II}	88.1664	92.5229	96.4136	101.1127
	$SBDH_T$	0.8412	0.9362	1.0298	1.1444
	$SBDH_B$	1.4808	1.6757	1.8609	2.0987

Table 1 (continued)
(c) Demeaned and detrended

n		90%	95%	97.5%	99%
n = 1	LM_I	0.2492	0.2498	0.2500	0.2500
	LM_{II}	9.5899	11.4628	13.2149	15.3448
	$SBDH_T$	0.0909	0.1107	0.1313	0.1588
	$SBDH_B$	0.1197	0.1478	0.1765	0.2173
n = 2	LM_I	0.6495	0.7544	0.8791	1.0606
	LM_{II}	20.9409	23.4075	25.6712	28.4783
	$SBDH_T$	0.1610	0.1864	0.2104	0.2414
	$SBDH_B$	0.2115	0.2476	0.2816	0.3297
n = 3	LM_I	1.1879	1.1347	1.5780	1.8635
	LM_{II}	35.7871	38.8298	41.5512	44.9975
	$SBDH_T$	0.2265	0.2549	0.2818	0.3171
	$SBDH_B$	0.2964	0.3359	0.3747	0.4261
n = 4	LM_I	1.8337	2.0938	2.3609	2.7603
	LM_{II}	54.1625	57.8637	61.0780	64.9712
	$SBDH_T$	0.2894	0.3208	0.3511	0.3893
	$SBDH_B$	0.3773	0.4220	0.4642	0.5189
n = 5	LM_I	2.5888	2.9310	3.2928	3.8092
	LM_{II}	76.3091	80.3840	84.2197	88.6674
	$SBDH_T$	0.3514	0.3858	0.4184	0.4573
	$SBDH_B$	0.4578	0.5068	0.5522	0.6114
n = 6	LM_I	3.4452	3.8830	4.3318	4.9508
	LM_{II}	102.2990	107.0575	111.1031	116.1178
	$SBDH_T$	0.4129	0.4500	0.4845	0.5249
	$SBDH_B$	0.5368	0.5884	0.6376	0.6979

1. Percentiles are obtained by FORTRAN from 100000 iteration.

Table 2 Empirical Size of LM_I, LM_{II}, and $SBDH$

$$B = \begin{bmatrix} 0.8 & 0.0 \\ 0.2 & 0.8 \end{bmatrix}, \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}, x_0 = 0, e_0 = 0.$$

(a) Standard

		Univariate tests			Multivariate tests		
		$l=2$	$l=l_1$	$l=l_2$	$l=2$	$l=l_1$	$l=l_2$
T = 100	LM_I	0.59	0.42	0.16	0.77	0.52	0.12
	LM_{II}	0.37	0.20	0.02	0.63	0.35	0.00
	$SBDH$	0.65	0.43	0.15	0.70	0.46	0.10
T = 200	LM_I	0.58	0.41	0.16	0.78	0.54	0.14
	LM_{II}	0.37	0.21	0.04	0.64	0.40	0.01
	$SBDH$	0.66	0.43	0.15	0.72	0.47	0.13
T = 400	LM_I	0.57	0.34	0.14	0.79	0.45	0.13
	LM_{II}	0.35	0.16	0.05	0.64	0.34	0.05
	$SBDH$	0.66	0.37	0.14	0.73	0.40	0.13
		(b) Demeaned					
T = 100	LM_I	0.27	0.11	0.07	0.82	0.57	0.06
	LM_{II}	0.00	0.00	0.00	0.37	0.14	0.00
	$SBDH_T$	0.86	0.54	0.11	0.92	0.60	0.07
	$SBDH_B$	0.79	0.51	0.17	0.85	0.56	0.12
T = 200	LM_I	0.29	0.15	0.07	0.84	0.65	0.09
	LM_{II}	0.01	0.00	0.00	0.44	0.24	0.00
	$SBDH_T$	0.90	0.60	0.15	0.95	0.68	0.12
	$SBDH_B$	0.82	0.53	0.18	0.88	0.60	0.14
T = 400	LM_I	0.32	0.14	0.09	0.87	0.59	0.11
	LM_{II}	0.02	0.00	0.01	0.48	0.23	0.01
	$SBDH_T$	0.92	0.55	0.17	0.96	0.60	0.13
	$SBDH_B$	0.84	0.47	0.17	0.90	0.51	0.14
		(c) Demeaned and Detrended					
T = 100	LM_I	0.06	0.05	0.06	0.76	0.54	0.01
	LM_{II}	0.00	0.00	0.00	0.30	0.07	0.00
	$SBDH_T$	0.95	0.66	0.11	0.98	0.74	0.09
	$SBDH_B$	0.94	0.67	0.18	0.97	0.74	0.14
T = 200	LM_I	0.11	0.05	0.06	0.82	0.62	0.05
	LM_{II}	0.00	0.00	0.00	0.40	0.20	0.00
	$SBDH_T$	0.98	0.75	0.15	0.99	0.82	0.11
	$SBDH_B$	0.96	0.71	0.18	0.98	0.79	0.15
T = 400	LM_I	0.14	0.06	0.07	0.84	0.58	0.07
	LM_{II}	0.00	0.00	0.00	0.45	0.20	0.00
	$SBDH_T$	0.99	0.71	0.19	1.00	0.76	0.15
	$SBDH_B$	0.98	0.66	0.21	0.99	0.71	0.16

Table 3 Empirical Power of LM_I, LM_{II}, and $SBDH$

$$B = \begin{bmatrix} 1.0 & 0.0 \\ 0.2 & 0.8 \end{bmatrix}, \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}, x_0 = 0, e_0 = 0.$$

(a) Standard

		Univariate tests			Multivariate tests		
		$l = 2$	$l = l_1$	$l = l_2$	$l = 2$	$l = l_1$	$l = l_2$
T = 100	LM_I	0.92	0.87	0.70	0.98	0.93	0.69
	LM_{II}	0.86	0.80	0.52	0.95	0.88	0.02
	$SBDH$	0.93	0.82	0.40	0.95	0.85	0.39
T = 200	LM_I	0.93	0.90	0.79	0.99	0.96	0.82
	LM_{II}	0.88	0.84	0.69	0.97	0.93	0.66
	$SBDH$	0.98	0.91	0.65	0.99	0.93	0.65
T = 400	LM_I	0.95	0.91	0.83	0.99	0.97	0.88
	LM_{II}	0.89	0.86	0.76	0.98	0.95	0.84
	$SBDH$	1.00	0.95	0.77	1.00	0.96	0.78
		(b) Demeaned					
T = 100	LM_I	0.74	0.56	0.02	0.95	0.85	0.22
	LM_{II}	0.14	0.00	0.00	0.41	0.13	0.02
	$SBDH_T$	0.99	0.91	0.51	1.00	0.95	0.49
	$SBDH_B$	0.98	0.91	0.66	0.99	0.94	0.65
T = 200	LM_I	0.82	0.72	0.24	0.98	0.94	0.47
	LM_{II}	0.44	0.15	0.00	0.73	0.39	0.01
	$SBDH_T$	1.00	0.99	0.71	1.00	0.99	0.73
	$SBDH_B$	1.00	0.98	0.77	1.00	0.99	0.79
T = 400	LM_I	0.88	0.78	0.54	0.99	0.96	0.77
	LM_{II}	0.59	0.34	0.00	0.88	0.65	0.08
	$SBDH_T$	1.00	1.00	0.88	1.00	1.00	0.88
	$SBDH_B$	1.00	0.99	0.89	1.00	1.00	0.89
		(c) Demeaned and Detrended					
T = 100	LM_I	0.28	0.02	0.03	0.83	0.58	0.05
	LM_{II}	0.00	0.00	0.00	0.27	0.07	0.00
	$SBDH_T$	0.99	0.90	0.34	1.00	0.95	0.34
	$SBDH_B$	1.00	0.93	0.52	1.00	0.96	0.51
T = 200	LM_I	0.55	0.29	0.03	0.94	0.82	0.26
	LM_{II}	0.03	0.00	0.00	0.47	0.24	0.00
	$SBDH_T$	1.00	0.99	0.61	1.00	1.00	0.62
	$SBDH_B$	1.00	0.99	0.73	1.00	0.99	0.73
T = 400	LM_I	0.69	0.46	0.02	0.98	0.90	0.45
	LM_{II}	0.20	0.00	0.00	0.71	0.40	0.03
	$SBDH_T$	1.00	1.00	0.86	1.00	1.00	0.86
	$SBDH_B$	1.00	1.00	0.90	1.00	1.00	0.90

Table 4 Empirical Power of LM_I, LM_{II}, and $SBDH$

$$B = \begin{bmatrix} 1.0 & 0.2 \\ 0.0 & 0.8 \end{bmatrix}, \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}, x_0 = 0, e_0 = 0.$$

		Univariate tests			Multivariate tests		
		$l=2$	$l=l_1$	$l=l_2$	$l=2$	$l=l_1$	$l=l_2$
			(a) Standard				
T = 100	LM_I	0.91	0.85	0.67	0.99	0.95	0.66
	LM_{II}	0.85	0.78	0.49	0.99	0.97	0.04
	$SBDH$	0.91	0.79	0.39	0.93	0.80	0.34
T = 200	LM_I	0.93	0.89	0.77	1.00	0.99	0.82
	LM_{II}	0.87	0.83	0.69	0.99	0.99	0.86
	$SBDH$	0.98	0.91	0.64	0.98	0.91	0.60
T = 400	LM_I	0.95	0.91	0.83	1.00	0.99	0.91
	LM_{II}	0.90	0.86	0.77	1.00	0.99	0.96
	$SBDH$	1.00	0.85	0.77	1.00	0.96	0.74
			(b) Demeaned				
T = 100	LM_I	0.77	0.59	0.05	1.00	0.97	0.33
	LM_{II}	0.16	0.00	0.00	0.78	0.40	0.00
	$SBDH_T$	0.99	0.93	0.51	1.00	0.95	0.46
	$SBDH_B$	0.99	0.93	0.69	0.99	0.94	0.63
T = 200	LM_I	0.85	0.74	0.26	1.00	1.00	0.66
	LM_{II}	0.45	0.14	0.00	0.97	0.81	0.04
	$SBDH_T$	1.00	0.99	0.73	1.00	0.99	0.71
	$SBDH_B$	1.00	0.98	0.79	1.00	0.99	0.78
T = 400	LM_I	0.89	0.79	0.55	1.00	1.00	0.93
	LM_{II}	0.60	0.35	0.00	0.99	0.95	0.35
	$SBDH_T$	1.00	1.00	0.88	1.00	1.00	0.86
	$SBDH_B$	1.00	0.99	0.89	1.00	0.99	0.87
			(c) Demeaned and Detrended				
T = 100	LM_I	0.33	0.04	0.05	0.98	0.89	0.05
	LM_{II}	0.00	0.00	0.00	0.75	0.33	0.00
	$SBDH_T$	1.00	0.92	0.36	1.00	0.94	0.26
	$SBDH_B$	1.00	0.95	0.56	1.00	0.96	0.42
T = 200	LM_I	0.56	0.31	0.05	1.00	0.99	0.43
	LM_{II}	0.03	0.00	0.00	0.95	0.78	0.00
	$SBDH_T$	1.00	0.99	0.63	1.00	0.99	0.56
	$SBDH_B$	1.00	0.99	0.74	1.00	0.99	0.66
T = 400	LM_I	0.69	0.48	0.05	1.00	1.00	0.78
	LM_{II}	0.20	0.01	0.00	0.99	0.93	0.24
	$SBDH_T$	1.00	1.00	0.86	1.00	1.00	0.81
	$SBDH_B$	1.00	1.00	0.90	1.00	1.00	0.85

1. Fraction of rejection from 5000 iteration each case.
2. Bartellet's kernel is used for estimating longrun variance.

Table 4 Percentiles for LM_I (Standard)

n	Percentile	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6
n = 1	80.0	0.2437	0.2441	0.2454	0.2463	0.2469	0.2474
	85.0	0.2472	0.2469	0.2475	0.2479	0.2483	0.2486
	90.0	0.2494	0.2488	0.2489	0.2491	0.2492	0.2494
	95.0	0.3036	0.2498	0.2498	0.2498	0.2498	0.2498
	97.5	0.9032	0.2500	0.2500	0.2499	0.2500	0.2500
	99.0	2.5445	0.5229	0.2500	0.2500	0.2500	0.2500
n = 2	80.0	0.8185	0.5920	0.5443	0.5273	0.5217	0.5172
	85.0	1.0160	0.6583	0.5793	0.5504	0.5404	0.5332
	90.0	1.4099	0.7709	0.6366	0.5890	0.5707	0.5576
	95.0	2.5155	1.0509	0.7626	0.6670	0.6312	0.6053
	97.5	4.3857	1.5717	0.9397	0.7691	0.7015	0.6584
	99.0	7.9024	2.9995	1.3049	0.9471	0.8165	0.7406
n = 3	80.0	2.0527	1.2207	1.0015	0.9202	0.8791	0.8529
	85.0	2.5480	1.3781	1.0770	0.9705	0.9178	0.8834
	90.0	3.4819	1.6494	1.1977	1.0468	0.9767	0.9284
	95.0	5.7062	2.3003	1.4780	1.1922	1.0830	1.0107
	97.5	8.9236	3.4278	1.8639	1.3845	1.2108	1.0957
	99.0	14.4986	5.7965	2.7544	1.7450	1.3934	1.2242
n = 4	80.0	3.8355	2.0743	1.5922	1.4042	1.3046	1.2419
	85.0	4.7504	2.3662	1.7229	1.4867	1.3657	1.2890
	90.0	6.3419	2.8721	1.9228	1.6090	1.4523	1.3558
	95.0	9.9709	4.0752	2.3978	1.8503	1.6156	1.4763
	97.5	14.4285	5.9130	3.0452	2.1346	1.7996	1.6048
	99.0	22.2499	9.3994	4.4329	2.6694	2.0937	1.8105
n = 5	80.0	6.1916	3.1531	2.3228	1.9744	1.7896	1.6825
	85.0	7.5904	3.5891	2.5313	2.0945	1.8713	1.7464
	90.0	9.9586	4.3568	2.8451	2.2716	1.9922	1.8405
	95.0	14.9707	6.1287	3.5457	2.6297	2.2144	2.0032
	97.5	21.3239	8.8480	4.4904	3.0591	2.4760	2.1839
	99.0	31.3771	13.8110	6.4955	3.8137	2.8955	2.4441
n = 6	80.0	9.1431	4.4892	3.1709	2.6312	2.3449	2.1778
	85.0	11.1061	5.1421	3.4510	2.7954	2.4543	2.2620
	90.0	14.3591	6.2806	3.9181	3.0431	2.6164	2.3787
	95.0	21.3865	8.9357	4.9132	3.5219	2.9141	2.5930
	97.5	29.8335	12.7621	6.2569	4.1417	3.2439	2.8134
	99.0	42.7340	19.1125	9.0353	5.3212	3.7998	3.1756

Table 6 Percentiles for LM_{II} (Standard)

n	Percentile	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6
n = 1	80.0	2.5709	3.4170	4.3106	5.1908	6.0450	6.8141
	85.0	3.1276	4.0910	5.0937	6.0357	6.9246	7.8061
	90.0	3.9040	5.0124	6.1394	7.1619	8.1586	9.0989
	95.0	5.2222	6.6258	7.9029	9.0001	10.1496	11.1987
	97.5	6.6127	8.1214	9.5620	10.7460	12.0665	13.1536
	99.0	8.4243	10.2593	11.6822	13.0848	14.4802	15.5838
n = 2	80.0	9.4273	10.8848	12.4319	14.0078	15.4867	16.9384
	85.0	10.3461	11.8686	13.4847	15.1192	16.6577	18.1699
	90.0	11.5575	13.1273	14.7980	16.5792	18.2060	19.7805
	95.0	13.5426	15.2007	17.0386	18.9842	20.6605	22.3786
	97.5	15.3095	17.1816	19.1432	21.2985	22.9823	24.7437
	99.0	17.6571	19.7355	21.8308	24.0450	25.9683	27.9961
n = 3	80.0	20.2925	22.2213	24.3578	26.5310	28.8783	31.0254
	85.0	21.5214	23.4987	25.6760	27.9599	30.3641	32.5787
	90.0	23.0755	25.1254	27.4498	29.7578	32.2528	34.5860
	95.0	25.5862	27.7373	30.2373	32.6519	35.2991	37.6136
	97.5	27.8607	30.1026	32.8490	35.2348	37.9976	40.4388
	99.0	30.8610	33.0612	35.9015	38.6289	41.8618	44.2828
n = 4	80.0	34.9778	37.3162	40.0581	42.9981	45.8813	48.7993
	85.0	36.5177	38.9013	41.6240	44.7357	47.6874	50.5531
	90.0	38.5407	40.9126	43.8259	46.9339	50.0516	52.9677
	95.0	41.6453	44.0221	47.2252	50.3192	53.6123	56.7808
	97.5	44.4768	46.9278	50.1980	53.4945	56.8813	59.9984
	99.0	47.8536	50.5164	53.7032	57.2386	60.6772	64.0372
n = 5	80.0	53.6717	56.3111	59.5939	62.8798	66.5900	70.2981
	85.0	55.4799	58.2347	61.5591	64.9822	68.6421	72.4531
	90.0	57.8932	60.6753	64.0381	67.6467	71.2566	75.1713
	95.0	61.6636	64.4687	67.9953	71.7425	75.4170	79.4815
	97.5	64.9452	67.9830	71.4581	75.3258	79.1816	83.2116
	99.0	69.1504	71.9643	75.7566	79.3927	83.6706	87.7385
n = 6	80.0	76.0539	79.2459	82.6764	86.6414	91.0903	95.2365
	85.0	78.2076	81.5085	84.9499	88.9537	93.5228	97.7492
	90.0	81.0740	84.3232	87.8220	91.9179	96.6445	100.9909
	95.0	85.3775	88.5987	92.2587	96.5143	101.3307	105.8249
	97.5	89.3121	92.5218	96.3610	100.6749	105.6705	109.9584
	99.0	93.7564	97.2678	100.9998	105.4917	110.2520	115.3793

Table 7 Percentiles for *SBDH* (Standard)

n	Percentile	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6
n = 1	80.0	0.5336	0.3902	0.2977	0.2385	0.1997	0.1711
	85.0	0.6614	0.4825	0.3673	0.2925	0.2445	0.2084
	90.0	0.8583	0.6268	0.4783	0.3738	0.3136	0.2664
	95.0	1.2132	0.8900	0.6788	0.5286	0.4430	0.3771
	97.5	1.5842	1.1826	0.9101	0.7075	0.5912	0.5021
	99.0	2.1317	1.5944	1.2289	0.9793	0.8034	0.6763
n = 2	80.0	1.0596	0.7742	0.5975	0.4780	0.3938	0.3351
	85.0	1.2368	0.9087	0.7007	0.5569	0.4551	0.3904
	90.0	1.4942	1.0993	0.8562	0.6717	0.5528	0.4690
	95.0	1.9458	1.4564	1.1242	0.8925	0.7284	0.6168
	97.5	2.4239	1.8185	1.4051	1.1201	0.9071	0.7772
	99.0	3.0732	2.3815	1.8361	1.4454	1.1728	0.9879
n = 3	80.0	1.5534	1.1428	0.8724	0.7027	0.5814	0.4942
	85.0	1.7710	1.3073	1.0020	0.8024	0.6625	0.5612
	90.0	2.0693	1.5496	1.1888	0.9502	0.7829	0.6570
	95.0	2.5946	1.9694	1.5215	1.2098	0.9999	0.8285
	97.5	3.1355	2.4070	1.8562	1.4866	1.2288	1.0054
	99.0	3.8814	3.0245	2.3363	1.8956	1.5757	1.2765
n = 4	80.0	2.0417	1.4982	1.1492	0.9173	0.7608	0.6443
	85.0	2.3001	1.6961	1.2943	1.0372	0.8571	0.7231
	90.0	2.6605	1.9671	1.5086	1.2088	1.0008	0.8345
	95.0	3.2325	2.4467	1.8862	1.5094	1.2418	1.0305
	97.5	3.7983	2.9220	2.2727	1.8328	1.5022	1.2457
	99.0	4.5727	3.5574	2.8083	2.2555	1.8581	1.5415
n = 5	80.0	2.4859	1.8430	1.4244	1.1328	0.9357	0.7961
	85.0	2.7714	2.0606	1.5967	1.2718	1.0430	0.8882
	90.0	3.1554	2.3663	1.8374	1.4640	1.1934	1.0174
	95.0	3.7984	2.9027	2.2595	1.7969	1.4670	1.2418
	97.5	4.4314	3.4314	2.6813	2.1606	1.7589	1.4801
	99.0	5.2790	4.1554	3.2809	2.6512	2.1651	1.7978
n = 6	80.0	2.9600	2.2001	1.6930	1.3405	1.1071	0.9405
	85.0	3.2640	2.4467	1.8866	1.4896	1.2246	1.0389
	90.0	3.6956	2.7886	2.1606	1.7059	1.3956	1.1798
	95.0	4.4002	3.3635	2.6228	2.0841	1.6975	1.4355
	97.5	5.0784	3.9272	3.0782	2.4888	2.0110	1.6930
	99.0	5.9630	4.6961	3.7107	3.0353	2.4191	2.0323

Table 8 Percentiles for LM_I (Demeaned)

n	Percentile	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6
n = 1	80.0	0.2465	0.2475	0.2481	0.2485	0.2487	0.2490
	85.0	0.2480	0.2486	0.2490	0.2492	0.2493	0.2494
	90.0	0.2491	0.2494	0.2495	0.2496	0.2497	0.2498
	95.0	0.2498	0.2498	0.2499	0.2499	0.2499	0.2499
	97.5	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500
	99.0	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500
n = 2	80.0	0.5646	0.5474	0.5362	0.5312	0.5260	0.5230
	85.0	0.6008	0.5731	0.5564	0.5479	0.5397	0.5352
	90.0	0.6569	0.6148	0.5869	0.5733	0.5610	0.5548
	95.0	0.7710	0.6942	0.6458	0.6222	0.6001	0.5900
	97.5	0.9127	0.7872	0.7108	0.6748	0.6463	0.6300
	99.0	1.1243	0.9336	0.8198	0.7551	0.7124	0.6843
n = 3	80.0	1.0370	0.9551	0.9064	0.8790	0.8592	0.8422
	85.0	1.1106	1.0058	0.9452	0.9100	0.8843	0.8647
	90.0	1.2191	1.0789	1.0006	0.9549	0.9202	0.8955
	95.0	1.4292	1.2194	1.1013	1.0362	0.9828	0.9498
	97.5	1.6672	1.3667	1.2156	1.1225	1.0517	1.0054
	99.0	2.0031	1.6022	1.3786	1.2487	1.1460	1.0799
n = 4	80.0	1.6240	1.4364	1.3322	1.2738	1.2275	1.1973
	85.0	1.7377	1.5149	1.3896	1.3180	1.2646	1.2280
	90.0	1.9084	1.6270	1.4742	1.3811	1.3180	1.2695
	95.0	2.1990	1.8263	1.6194	1.4935	1.4090	1.3443
	97.5	2.5284	2.0515	1.7725	1.6080	1.5030	1.4212
	99.0	3.0437	2.3755	1.9968	1.7787	1.6285	1.5191
n = 5	80.0	2.3226	2.0013	1.8230	1.7106	1.6355	1.5830
	85.0	2.4842	2.1055	1.8994	1.7715	1.6833	1.6226
	90.0	2.7126	2.2571	2.0067	1.8554	1.7501	1.6789
	95.0	3.1382	2.5344	2.1855	2.0024	1.8666	1.7711
	97.5	3.6327	2.8318	2.3931	2.1545	1.9753	1.8686
	99.0	4.3553	3.3034	2.6802	2.3681	2.1292	2.0022
n = 6	80.0	3.1506	2.6471	2.3563	2.1884	2.0818	1.9969
	85.0	3.3623	2.7820	2.4521	2.2637	2.1410	2.0472
	90.0	3.6677	2.9787	2.5946	2.3691	2.2264	2.1175
	95.0	4.2250	3.3305	2.8405	2.5449	2.3718	2.2317
	97.5	4.8285	3.7292	3.0852	2.7206	2.5116	2.3445
	99.0	5.7534	4.3056	3.4359	2.9756	2.7045	2.4999

Table 9 Percentiles for LM_{II} (Demeaned)

n	Percentile	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6
n = 1	80.0	7.1762	9.6126	11.9301	14.2269	16.5973	18.7670
	85.0	8.0316	10.5198	12.9698	15.3275	17.7321	20.0164
	90.0	9.1525	11.7931	14.3561	16.7887	19.2769	21.6864
	95.0	10.9099	13.8092	16.6086	19.0831	21.7412	24.2678
	97.5	12.6799	15.5878	18.6422	21.1779	24.0997	26.5259
	99.0	14.9562	17.9105	21.1685	23.8297	26.9834	29.4622
n = 2	80.0	17.3303	21.7721	26.1513	30.4971	34.7387	39.1603
	85.0	18.5145	23.1510	27.6003	32.0995	36.3710	40.8938
	90.0	20.0872	24.8577	29.5085	34.1196	38.4864	43.1702
	95.0	22.5893	27.5014	32.3490	37.2962	41.7797	46.7142
	97.5	24.8841	29.9888	35.0361	40.1632	44.8102	49.6717
	99.0	27.6102	33.0736	38.2635	43.6378	48.3947	53.5310
n = 3	80.0	31.1272	37.4997	43.8191	50.2451	56.5445	62.7026
	85.0	32.6274	39.2213	45.6021	52.1526	58.5455	64.9128
	90.0	34.6752	41.4005	47.9422	54.5987	61.1815	67.6094
	95.0	37.6457	44.6952	51.5388	58.4182	65.1352	71.8293
	97.5	40.4527	47.8406	54.7544	61.8523	68.8149	75.7351
	99.0	43.8557	51.4284	58.8092	66.1747	73.2717	80.1752
n = 4	80.0	48.7143	57.0225	65.2570	73.5334	81.6219	89.9856
	85.0	50.5938	59.0167	67.3497	75.9159	84.0330	92.4201
	90.0	52.9798	61.6339	70.2033	78.8175	87.2199	95.7463
	95.0	56.5722	65.5914	74.4239	83.2953	91.8193	100.4379
	97.5	59.6847	69.2061	78.1617	87.0748	96.2869	104.6842
	99.0	63.6372	73.3739	82.7422	91.6949	101.3816	109.8979
n = 5	80.0	70.2398	80.5093	90.5150	100.5141	110.7268	120.7246
	85.0	72.3576	82.8615	92.9578	103.1082	113.4694	123.5394
	90.0	75.0910	85.7999	96.0979	106.3955	116.9811	127.1228
	95.0	79.4220	90.2783	100.9179	111.4465	122.3454	132.6250
	97.5	83.0968	94.2468	105.1320	115.8080	126.8255	137.6890
	99.0	87.7802	99.0079	110.1166	121.2237	132.5819	143.2921
n = 6	80.0	95.5274	107.6418	119.5446	131.4923	143.2641	155.2408
	85.0	98.0206	110.2411	122.3393	134.3788	146.3465	158.4407
	90.0	101.1618	113.6969	125.8283	138.0724	150.2357	162.3995
	95.0	106.0511	118.7723	131.2636	143.6167	156.1188	168.6000
	97.5	110.4545	123.2418	135.9532	148.5755	161.1717	174.0185
	99.0	115.7110	128.4192	142.3224	154.6084	167.5370	180.7248

Table 10 Percentiles for $SBDH_I$ (Demeaned)

n	Percentile	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6
n = 1	80.0	0.0845	0.0563	0.0420	0.0333	0.0272	0.0231
	85.0	0.0961	0.0631	0.0466	0.0365	0.0295	0.0249
	90.0	0.1133	0.0728	0.0530	0.0409	0.0329	0.0275
	95.0	0.1460	0.0909	0.0647	0.0491	0.0385	0.0321
	97.5	0.1830	0.1107	0.0770	0.0574	0.0444	0.0369
	99.0	0.2389	0.1422	0.0962	0.0699	0.0531	0.0435
n = 2	80.0	0.1625	0.1094	0.0802	0.0633	0.0522	0.0443
	85.0	0.1794	0.1196	0.0867	0.0677	0.0554	0.0468
	90.0	0.2037	0.1334	0.0958	0.0742	0.0600	0.0504
	95.0	0.2481	0.1585	0.1118	0.0852	0.0676	0.0563
	97.5	0.2961	0.1855	0.1287	0.0963	0.0753	0.0623
	99.0	0.3615	0.2260	0.1527	0.1110	0.0861	0.0706
n = 3	80.0	0.2367	0.1596	0.1179	0.0930	0.0762	0.0646
	85.0	0.2577	0.1714	0.1257	0.0986	0.0801	0.0678
	90.0	0.2878	0.1888	0.1368	0.1065	0.0856	0.0719
	95.0	0.3424	0.2200	0.1565	0.1196	0.0944	0.0789
	97.5	0.4009	0.2524	0.1767	0.1334	0.1034	0.0859
	99.0	0.4815	0.2996	0.2048	0.1539	0.1152	0.0959
n = 4	80.0	0.3089	0.2085	0.1542	0.1217	0.1002	0.0848
	85.0	0.3340	0.2231	0.1633	0.1278	0.1050	0.0883
	90.0	0.3702	0.2438	0.1763	0.1366	0.1112	0.0931
	95.0	0.4329	0.2806	0.1985	0.1520	0.1215	0.1011
	97.5	0.5007	0.3225	0.2215	0.1671	0.1320	0.1088
	99.0	0.5946	0.3806	0.2536	0.1890	0.1466	0.1194
n = 5	80.0	0.3823	0.2565	0.1903	0.1502	0.1236	0.1052
	85.0	0.4114	0.2728	0.2005	0.1575	0.1287	0.1091
	90.0	0.4530	0.2961	0.2145	0.1675	0.1358	0.1147
	95.0	0.5278	0.3366	0.2393	0.1839	0.1475	0.1235
	97.5	0.6028	0.3798	0.2665	0.2008	0.1591	0.1324
	99.0	0.7051	0.4471	0.3049	0.2241	0.1751	0.1446
n = 6	80.0	0.4557	0.3057	0.2270	0.1782	0.1471	0.1248
	85.0	0.4889	0.3236	0.2388	0.1861	0.1528	0.1292
	90.0	0.5362	0.3499	0.2545	0.1970	0.1606	0.1352
	95.0	0.6183	0.3963	0.2835	0.2163	0.1743	0.1449
	97.5	0.6988	0.4445	0.3118	0.2357	0.1873	0.1546
	99.0	0.8021	0.5159	0.3559	0.2617	0.2057	0.1679

Table 11 Percentiles for $SBDH_{II}$ (Demeaned)

n	Percentile	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6
n = 1	80.0	0.1605	0.1150	0.0879	0.0706	0.0587	0.0500
	85.0	0.1892	0.1337	0.1013	0.0803	0.0657	0.0558
	90.0	0.2321	0.1619	0.1209	0.0949	0.0763	0.0642
	95.0	0.3159	0.2195	0.1602	0.1225	0.0969	0.0801
	97.5	0.4079	0.2863	0.2046	0.1525	0.1208	0.0977
	99.0	0.5348	0.3927	0.2738	0.2029	0.1579	0.1250
n = 2	80.0	0.3162	0.2259	0.1718	0.1365	0.1127	0.0955
	85.0	0.3581	0.2545	0.1920	0.1508	0.1235	0.1039
	90.0	0.4216	0.2979	0.2221	0.1715	0.1392	0.1161
	95.0	0.5312	0.3764	0.2782	0.2103	0.1685	0.1386
	97.5	0.6415	0.4662	0.3417	0.2558	0.2013	0.1638
	99.0	0.7963	0.5898	0.4360	0.3219	0.2509	0.2015
n = 3	80.0	0.4713	0.3336	0.2525	0.2002	0.1656	0.1403
	85.0	0.5249	0.3709	0.2776	0.2187	0.1794	0.1510
	90.0	0.6053	0.4250	0.3163	0.2461	0.1996	0.1659
	95.0	0.7368	0.5218	0.3864	0.2970	0.2359	0.1939
	97.5	0.8789	0.6272	0.4645	0.3569	0.2796	0.2246
	99.0	1.0516	0.7825	0.5826	0.4426	0.3435	0.2692
n = 4	80.0	0.6115	0.4380	0.3323	0.2638	0.2175	0.1844
	85.0	0.6754	0.4837	0.3629	0.2860	0.2341	0.1972
	90.0	0.7648	0.5486	0.4083	0.3183	0.2588	0.2157
	95.0	0.9172	0.6651	0.4910	0.3786	0.3017	0.2483
	97.5	1.0685	0.7863	0.5777	0.4454	0.3514	0.2831
	99.0	1.2657	0.9513	0.7126	0.5518	0.4198	0.3345
n = 5	80.0	0.7604	0.5405	0.4116	0.3265	0.2686	0.2283
	85.0	0.8305	0.5932	0.4473	0.3523	0.2880	0.2434
	90.0	0.9275	0.6692	0.4989	0.3901	0.3150	0.2643
	95.0	1.0997	0.7998	0.5985	0.4597	0.3667	0.3018
	97.5	1.2694	0.9391	0.7071	0.5356	0.4194	0.3437
	99.0	1.4863	1.1275	0.8516	0.6525	0.5040	0.4019
n = 6	80.0	0.9034	0.6451	0.4906	0.3878	0.3191	0.2711
	85.0	0.9831	0.7059	0.5314	0.4174	0.3406	0.2875
	90.0	1.0915	0.7928	0.5915	0.4595	0.3717	0.3115
	95.0	1.2773	0.9419	0.7022	0.5378	0.4278	0.3536
	97.5	1.4511	1.0961	0.8184	0.6225	0.4872	0.3992
	99.0	1.6733	1.3028	0.9863	0.7532	0.5826	0.4725

Table 12 Percentiles for LM_I (Detrended)

n	Percentile	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6
n = 1	80.0	0.2477	0.2482	0.2486	0.2487	0.2489	0.2491
	85.0	0.2487	0.2490	0.2492	0.2493	0.2494	0.2495
	90.0	0.2494	0.2495	0.2496	0.2497	0.2497	0.2498
	95.0	0.2499	0.2499	0.2499	0.2499	0.2499	0.2499
	97.5	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500
	99.0	0.2500	0.2500	0.2500	0.2500	0.2500	0.2500
n = 2	80.0	0.5435	0.5351	0.5306	0.5262	0.5230	0.5201
	85.0	0.5679	0.5547	0.5468	0.5396	0.5349	0.5307
	90.0	0.6058	0.5858	0.5724	0.5607	0.5537	0.5471
	95.0	0.6826	0.6446	0.6210	0.5992	0.5881	0.5774
	97.5	0.7684	0.7090	0.6746	0.6443	0.6283	0.6106
	99.0	0.9010	0.8055	0.7512	0.7122	0.6869	0.6597
n = 3	80.0	0.9401	0.9025	0.8764	0.8587	0.8434	0.8336
	85.0	0.9858	0.9402	0.9071	0.8842	0.8650	0.8527
	90.0	1.0534	0.9944	0.9503	0.9213	0.8947	0.8802
	95.0	1.1768	1.0931	1.0291	0.9859	0.9478	0.9268
	97.5	1.3177	1.1999	1.1136	1.0512	1.0063	0.9742
	99.0	1.5234	1.3543	1.2307	1.1501	1.0875	1.0479
n = 4	80.0	1.4091	1.3269	1.2660	1.2257	1.1948	1.1721
	85.0	1.4790	1.3815	1.3098	1.2613	1.2248	1.1982
	90.0	1.5791	1.4605	1.3704	1.3120	1.2678	1.2355
	95.0	1.7590	1.6006	1.4812	1.4008	1.3437	1.2996
	97.5	1.9501	1.7443	1.5895	1.4896	1.4203	1.3632
	99.0	2.2190	1.9459	1.7404	1.6154	1.5189	1.4455
n = 5	80.0	1.9459	1.8015	1.7037	1.6332	1.5817	1.5386
	85.0	2.0395	1.8761	1.7630	1.6812	1.6227	1.5728
	90.0	2.1755	1.9821	1.8437	1.7464	1.6778	1.6215
	95.0	2.4129	2.1618	1.9837	1.8638	1.7684	1.7022
	97.5	2.6591	2.3461	2.1263	1.9799	1.8647	1.7850
	99.0	3.0234	2.6319	2.3200	2.1335	1.9926	1.9018
n = 6	80.0	2.5555	2.3359	2.1796	2.0769	1.9950	1.9366
	85.0	2.6765	2.4314	2.2525	2.1380	2.0464	1.9796
	90.0	2.8519	2.5633	2.3533	2.2204	2.1161	2.0368
	95.0	3.1547	2.7854	2.5260	2.3606	2.2303	2.1345
	97.5	3.4589	3.0247	2.7001	2.5024	2.3426	2.2338
	99.0	3.9208	3.3568	2.9564	2.6841	2.4929	2.3584

Table 13 Percentiles for LM_{II} (Detrended)

n	Percentile	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6
n = 1	80.0	9.8129	12.0949	14.3799	16.5486	18.8354	21.1028
	85.0	10.7570	13.1239	15.5119	17.7313	20.0758	22.4270
	90.0	12.0415	14.4712	16.9648	19.3072	21.6749	24.0910
	95.0	14.0315	16.6214	19.2463	21.7245	24.2017	26.7028
	97.5	15.8423	18.4932	21.4231	23.9274	26.6642	29.2167
	99.0	18.3082	21.0856	24.0219	26.6802	29.6105	32.1198
n = 2	80.0	22.1206	26.3875	30.6943	35.0177	39.2347	43.4898
	85.0	23.4311	27.8350	32.2162	36.6614	40.9854	45.3553
	90.0	25.1862	29.7988	34.2189	38.8068	43.2710	47.6533
	95.0	27.9566	32.6699	37.4264	42.1717	46.8651	51.3063
	97.5	30.4405	35.4499	40.1859	45.1936	50.0110	54.6356
	99.0	33.6920	39.0203	44.0434	48.8252	53.7368	58.5367
n = 3	80.0	38.0286	44.2091	50.4044	56.8008	62.8464	68.9902
	85.0	39.6559	45.9733	52.3439	58.8194	64.9367	71.1540
	90.0	41.9000	48.2886	54.8801	61.3978	67.7452	74.0625
	95.0	45.1838	51.8177	58.7027	65.3096	71.8894	78.4133
	97.5	48.2195	55.1547	62.2333	68.9340	75.8450	82.2480
	99.0	51.9578	59.4050	66.2043	73.1684	80.2230	86.8722
n = 4	80.0	57.5533	65.7962	73.6772	81.8823	90.0577	98.2578
	85.0	59.5573	67.9170	76.0172	84.2253	92.5525	100.8581
	90.0	62.0884	70.6629	78.9114	87.3308	95.7163	104.0957
	95.0	65.9285	74.9775	83.3934	92.0733	100.5286	109.1300
	97.5	69.3574	78.7374	87.5143	96.1099	104.7812	113.6196
	99.0	73.7297	82.9909	92.6478	100.8058	110.0271	118.7033
n = 5	80.0	80.9272	90.7925	100.9372	111.0018	120.8225	130.9411
	85.0	83.2726	93.1670	103.4809	113.6661	123.6436	133.8571
	90.0	86.2079	96.2016	106.7626	117.2038	127.2677	137.6355
	95.0	90.6998	101.1296	111.8444	122.7103	132.7901	143.1030
	97.5	94.6788	105.5537	116.2289	127.2147	137.6111	148.2073
	99.0	99.5905	110.7282	121.6138	132.6195	143.4756	154.1548
n = 6	80.0	107.8572	119.6488	131.7345	143.6312	155.6095	167.2452
	85.0	110.4673	122.3205	134.5998	146.5380	158.6935	170.4946
	90.0	113.8278	125.8779	138.2493	150.5703	162.6876	174.6662
	95.0	118.8295	131.1250	143.7042	156.4024	168.6027	180.9152
	97.5	123.2913	135.9239	148.7574	161.3756	173.8214	186.5704
	99.0	128.7368	141.3892	155.2307	167.4478	180.3714	193.4725

Table 14 Percentiles for $SBDH_I$ (Detrended)

n	Percentile	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6
n = 1	80.0	0.0533	0.0411	0.0328	0.0273	0.0230	0.0200
	85.0	0.0592	0.0452	0.0359	0.0295	0.0248	0.0215
	90.0	0.0674	0.0510	0.0401	0.0327	0.0274	0.0234
	95.0	0.0820	0.0613	0.0477	0.0385	0.0318	0.0270
	97.5	0.0982	0.0728	0.0557	0.0441	0.0363	0.0305
	99.0	0.1201	0.0891	0.0676	0.0524	0.0430	0.0356
n = 2	80.0	0.1019	0.0782	0.0625	0.0518	0.0438	0.0381
	85.0	0.1102	0.0840	0.0667	0.0550	0.0463	0.0401
	90.0	0.1216	0.0925	0.0724	0.0595	0.0496	0.0430
	95.0	0.1403	0.1063	0.0828	0.0669	0.0554	0.0476
	97.5	0.1598	0.1207	0.0933	0.0747	0.0615	0.0522
	99.0	0.1849	0.1417	0.1086	0.0853	0.0696	0.0585
n = 3	80.0	0.1481	0.1144	0.0917	0.0756	0.0643	0.0558
	85.0	0.1574	0.1215	0.0969	0.0797	0.0673	0.0583
	90.0	0.1713	0.1312	0.1041	0.0850	0.0715	0.0616
	95.0	0.1932	0.1483	0.1164	0.0944	0.0783	0.0674
	97.5	0.2158	0.1658	0.1283	0.1028	0.0852	0.0727
	99.0	0.2461	0.1877	0.1448	0.1159	0.0949	0.0798
n = 4	80.0	0.1935	0.1494	0.1201	0.0994	0.0844	0.0732
	85.0	0.2050	0.1575	0.1261	0.1040	0.0879	0.0760
	90.0	0.2200	0.1691	0.1344	0.1102	0.0927	0.0797
	95.0	0.2459	0.1888	0.1484	0.1204	0.1007	0.0858
	97.5	0.2716	0.2090	0.1619	0.1308	0.1085	0.0920
	99.0	0.3012	0.2360	0.1803	0.1450	0.1188	0.1002
n = 5	80.0	0.2385	0.1847	0.1479	0.1224	0.1043	0.0906
	85.0	0.2516	0.1942	0.1547	0.1275	0.1082	0.0937
	90.0	0.2688	0.2069	0.1639	0.1344	0.1134	0.0978
	95.0	0.2978	0.2285	0.1796	0.1457	0.1221	0.1048
	97.5	0.3252	0.2509	0.1953	0.1571	0.1310	0.1117
	99.0	0.3615	0.2795	0.2176	0.1730	0.1421	0.1201
n = 6	80.0	0.2832	0.2198	0.1753	0.1457	0.1238	0.1077
	85.0	0.2973	0.2301	0.1829	0.1515	0.1283	0.1112
	90.0	0.3164	0.2443	0.1932	0.1591	0.1341	0.1158
	95.0	0.3469	0.2674	0.2103	0.1716	0.1438	0.1233
	97.5	0.3780	0.2908	0.2278	0.1839	0.1532	0.1307
	99.0	0.4147	0.3224	0.2521	0.2022	0.1666	0.1402

Table 15 Percentiles for $SBDH_{II}$ (Detrended)

n	Percentile	m = 1	m = 2	m = 3	m = 4	m = 5	m = 6
n = 1	80.0	0.0761	0.0637	0.0544	0.0475	0.0416	0.0370
	85.0	0.0853	0.0713	0.0606	0.0525	0.0460	0.0407
	90.0	0.0986	0.0822	0.0693	0.0598	0.0520	0.0459
	95.0	0.1222	0.1024	0.0858	0.0735	0.0631	0.0551
	97.5	0.1476	0.1237	0.1036	0.0870	0.0750	0.0650
	99.0	0.1812	0.1533	0.1271	0.1071	0.0915	0.0795
n = 2	80.0	0.1454	0.1219	0.1042	0.0904	0.0793	0.0707
	85.0	0.1581	0.1325	0.1128	0.0975	0.0850	0.0758
	90.0	0.1759	0.1473	0.1251	0.1077	0.0936	0.0829
	95.0	0.2068	0.1737	0.1468	0.1248	0.1085	0.0958
	97.5	0.2368	0.1995	0.1687	0.1444	0.1236	0.1084
	99.0	0.2779	0.2351	0.1997	0.1703	0.1463	0.1272
n = 3	80.0	0.2108	0.1784	0.1527	0.1324	0.1163	0.1032
	85.0	0.2265	0.1916	0.1632	0.1410	0.1237	0.1096
	90.0	0.2473	0.2095	0.1778	0.1534	0.1341	0.1186
	95.0	0.2818	0.2395	0.2039	0.1751	0.1514	0.1338
	97.5	0.3163	0.2693	0.2290	0.1966	0.1699	0.1498
	99.0	0.3634	0.3145	0.2641	0.2279	0.1964	0.1726
n = 4	80.0	0.2757	0.2330	0.1997	0.1739	0.1525	0.1356
	85.0	0.2937	0.2483	0.2121	0.1845	0.1611	0.1430
	90.0	0.3176	0.2690	0.2291	0.1995	0.1729	0.1536
	95.0	0.3585	0.3039	0.2580	0.2245	0.1940	0.1709
	97.5	0.3964	0.3370	0.2866	0.2489	0.2140	0.1890
	99.0	0.4476	0.3818	0.3255	0.2855	0.2425	0.2124
n = 5	80.0	0.3398	0.2879	0.2461	0.2139	0.1883	0.1678
	85.0	0.3596	0.3049	0.2604	0.2260	0.1982	0.1760
	90.0	0.3863	0.3283	0.2796	0.2419	0.2123	0.1877
	95.0	0.4295	0.3675	0.3124	0.2694	0.2351	0.2071
	97.5	0.4726	0.4043	0.3443	0.2968	0.2575	0.2261
	99.0	0.5265	0.4508	0.3878	0.3352	0.2884	0.2532
n = 6	80.0	0.4031	0.3424	0.2926	0.2551	0.2237	0.1992
	85.0	0.4254	0.3617	0.3084	0.2682	0.2348	0.2087
	90.0	0.4539	0.3866	0.3299	0.2867	0.2502	0.2214
	95.0	0.5003	0.4283	0.3660	0.3167	0.2755	0.2431
	97.5	0.5442	0.4695	0.4031	0.3470	0.3012	0.2654
	99.0	0.6060	0.5209	0.4473	0.3871	0.3381	0.2956

1. Percentiles are obtained by GAUSS from 100000 iteration.
2. m denotes number of $I(1)$ regressors.

Table 16 Empirical Size of LM_I, LM_{II}, and $SBDH$

$$DGP: u_t = \begin{bmatrix} 0.8 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} u_{t-1} + \epsilon_{1t}, \Delta x_t = \epsilon_{2t}, \epsilon_t \equiv iidN(0, \Sigma).$$

		Single Equation Test	System of Equations Test
		(a) Standard	
T = 100	LM_I	0.14	0.28
	LM_{II}	0.02	0.02
	$SBDH$	0.23	0.28
T = 200	LM_I	0.11	0.26
	LM_{II}	0.02	0.02
	$SBDH$	0.19	0.23
T = 400	LM_I	0.09	0.24
	LM_{II}	0.03	0.04
	$SBDH$	0.16	0.19
		(b) Demeaned	
T = 100	LM_I	0.00	0.05
	LM_{II}	0.00	0.00
	$SBDH_T$	0.11	0.17
	$SBDH_B$	0.26	0.33
T = 200	LM_I	0.00	0.07
	LM_{II}	0.00	0.01
	$SBDH_T$	0.09	0.09
	$SBDH_B$	0.21	0.25
T = 400	LM_I	0.01	0.10
	LM_{II}	0.01	0.02
	$SBDH_T$	0.06	0.06
	$SBDH_B$	0.16	0.21
		(b) Demeaned and Detrended	
T = 100	LM_I	0.00	0.05
	LM_{II}	0.00	0.00
	$SBDH_T$	0.20	0.38
	$SBDH_B$	0.36	0.52
T = 200	LM_I	0.00	0.09
	LM_{II}	0.00	0.01
	$SBDH_T$	0.15	0.20
	$SBDH_B$	0.18	0.37
T = 400	LM_I	0.00	0.11
	LM_{II}	0.00	0.02
	$SBDH_T$	0.09	0.10
	$SBDH_B$	0.19	0.27

Table 17 Empirical Power of LM_I, LM_{II}, and $SBDH$

$$DGP: u_t = \begin{bmatrix} 1.0 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} u_{t-1} + \epsilon_{1t}, \Delta x_t = \epsilon_{2t}, \epsilon_t \equiv iidN(0, \Sigma).$$

		Single Equation Test	System of Equations Test
		(a) Standard	
T = 100	LM_I	0.41	0.59
	LM_{II}	0.12	0.23
	$SBDH$	0.54	0.54
T = 200	LM_I	0.50	0.71
	LM_{II}	0.18	0.43
	$SBDH$	0.70	0.67
T = 400	LM_I	0.64	0.86
	LM_{II}	0.26	0.67
	$SBDH$	0.88	0.84
		(b) Demeaned	
T = 100	LM_I	0.02	0.15
	LM_{II}	0.00	0.00
	$SBDH_T$	0.38	0.41
	$SBDH_B$	0.66	0.62
T = 200	LM_I	0.13	0.44
	LM_{II}	0.00	0.08
	$SBDH_T$	0.69	0.59
	$SBDH_B$	0.84	0.75
T = 400	LM_I	0.31	0.76
	LM_{II}	0.03	0.41
	$SBDH_T$	0.91	0.83
	$SBDH_B$	0.96	0.91
		(b) Demeaned and Detrended	
T = 100	LM_I	0.00	0.11
	LM_{II}	0.00	0.00
	$SBDH_T$	0.39	0.54
	$SBDH_B$	0.68	0.74
T = 200	LM_I	0.03	0.37
	LM_{II}	0.00	0.05
	$SBDH_T$	0.68	0.59
	$SBDH_B$	0.85	0.79
T = 400	LM_I	0.17	0.75
	LM_{II}	0.00	0.36
	$SBDH_T$	0.92	0.83
	$SBDH_B$	0.96	0.90

Table 18 Empirical Size of LM_I, LM_{II}, and $SBDH$

$$DGP: u_t = \begin{bmatrix} 1.0 & 0.2 \\ 0.0 & 0.8 \end{bmatrix} u_{t-1} + \epsilon_{1t}, \Delta x_t = \epsilon_{2t}, \epsilon_t \equiv iidN(0, \Sigma).$$

		Single Equation Test	System of Equations Test
		(a) Standard	
T = 100	LM_I	0.38	0.58
	LM_{II}	0.12	0.21
	$SBDH$	0.51	0.54
T = 200	LM_I	0.50	0.73
	LM_{II}	0.17	0.42
	$SBDH$	0.69	0.68
T = 400	LM_I	0.64	0.87
	LM_{II}	0.25	0.68
	$SBDH$	0.88	0.85
		(b) Demeaned	
T = 100	LM_I	0.02	0.14
	LM_{II}	0.00	0.00
	$SBDH_T$	0.33	0.41
	$SBDH_B$	0.62	0.62
T = 200	LM_I	0.12	0.44
	LM_{II}	0.00	0.08
	$SBDH_T$	0.66	0.60
	$SBDH_B$	0.84	0.76
T = 400	LM_I	0.31	0.76
	LM_{II}	0.03	0.40
	$SBDH_T$	0.90	0.83
	$SBDH_B$	0.96	0.91
		(b) Demeaned and Detrended	
T = 100	LM_I	0.00	0.09
	LM_{II}	0.00	0.00
	$SBDH_T$	0.37	0.60
	$SBDH_B$	0.66	0.80
T = 200	LM_I	0.03	0.36
	LM_{II}	0.00	0.05
	$SBDH_T$	0.65	0.65
	$SBDH_B$	0.84	0.84
T = 400	LM_I	0.16	0.74
	LM_{II}	0.00	0.33
	$SBDH_T$	0.92	0.87
	$SBDH_B$	0.97	0.94

1. Fraction of rejection out of 5000 iterations.
2. Quadratic kernel is used for long run variance estimation.

Table 19 Empirical Percentiles (Model 1)

λ	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
$\lambda = 0.25$	0.9000	0.2491	9.5635	0.0910	0.2091
	0.9500	0.2498	11.4752	0.1113	0.2720
	0.9750	0.2499	13.0953	0.1337	0.3339
	0.9900	0.2500	15.3200	0.1630	0.4285
$\lambda = 0.33$	0.9000	0.2492	9.6858	0.0876	0.1806
	0.9500	0.2498	11.6306	0.1074	0.2303
	0.9750	0.2500	13.3553	0.1284	0.2738
	0.9900	0.2500	15.6005	0.1568	0.3447
$\lambda = 0.41$	0.9000	0.2492	9.6388	0.0887	0.1593
	0.9500	0.2498	11.5143	0.1096	0.1977
	0.9750	0.2499	13.2342	0.1299	0.2373
	0.9900	0.2500	15.2807	0.1589	0.2953
$\lambda = 0.49$	0.9000	0.2492	9.5871	0.0936	0.1553
	0.9500	0.2498	11.3977	0.1155	0.1913
	0.9750	0.2500	13.1855	0.1393	0.2283
	0.9900	0.2500	15.1990	0.1729	0.2718
$\lambda = 0.59$	0.9000	0.2494	9.3981	0.1043	0.1615
	0.9500	0.2498	11.2113	0.1329	0.2048
	0.9750	0.2500	13.0874	0.1624	0.2470
	0.9900	0.2500	15.0692	0.2032	0.3120
$\lambda = 0.63$	0.9000	0.2493	9.2139	0.1114	0.1692
	0.9500	0.2498	11.1312	0.1402	0.2175
	0.9750	0.2500	12.8281	0.1743	0.2695
	0.9900	0.2500	15.0348	0.2164	0.3352

Table 20 Empirical Percentiles (Model 2)

λ	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
$\lambda = 0.25$	0.9000	0.2495	12.5206	0.0575	0.0846
	0.9500	0.2499	14.5150	0.0689	0.1025
	0.9750	0.2500	16.3454	0.0794	0.1201
	0.9900	0.2500	18.7922	0.0931	0.1425
$\lambda = 0.33$	0.9000	0.2495	12.2511	0.0625	0.0888
	0.9500	0.2499	14.3290	0.0753	0.1071
	0.9750	0.2500	16.3372	0.0884	0.1273
	0.9900	0.2500	18.7455	0.1092	0.1506
$\lambda = 0.41$	0.9000	0.2495	11.8274	0.0753	0.0992
	0.9500	0.2499	13.9307	0.0941	0.1260
	0.9750	0.2500	15.6970	0.1162	0.1518
	0.9900	0.2500	18.0947	0.1451	0.1903
$\lambda = 0.49$	0.9000	0.2491	11.6614	0.0742	0.1059
	0.9500	0.2498	13.6926	0.0931	0.1356
	0.9750	0.2499	15.7077	0.1116	0.1653
	0.9900	0.2500	17.8886	0.1342	0.2120
$\lambda = 0.59$	0.9000	0.2493	12.3129	0.0600	0.1006
	0.9500	0.2498	14.3560	0.0716	0.1253
	0.9750	0.2500	16.4527	0.0830	0.1527
	0.9900	0.2500	18.7032	0.0972	0.1842
$\lambda = 0.63$	0.9000	0.2494	12.4516	0.0582	0.0954
	0.9500	0.2498	14.4078	0.0689	0.1176
	0.9750	0.2500	16.5541	0.0809	0.1379
	0.9900	0.2500	18.8471	0.0944	0.1651

Table 21 Empirical Percentiles (Model 3)

λ	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
$\lambda = 0.25$	0.9000	0.2494	12.5751	0.0596	0.0845
	0.9500	0.2498	14.6485	0.0715	0.1041
	0.9750	0.2500	16.5370	0.0820	0.1232
	0.9900	0.2500	19.2334	0.0978	0.1484
$\lambda = 0.33$	0.9000	0.2494	12.7024	0.0572	0.0762
	0.9500	0.2499	14.8097	0.0686	0.0937
	0.9750	0.2500	16.7931	0.0797	0.1111
	0.9900	0.2500	19.0340	0.0926	0.1326
$\lambda = 0.41$	0.9000	0.2495	12.7977	0.0566	0.0718
	0.9500	0.2499	14.8648	0.0674	0.0864
	0.9750	0.2500	16.7080	0.0787	0.1031
	0.9900	0.2500	19.1433	0.0927	0.1227
$\lambda = 0.49$	0.9000	0.2494	12.7212	0.0578	0.0698
	0.9500	0.2499	14.7501	0.0683	0.0843
	0.9750	0.2500	16.7291	0.0799	0.0979
	0.9900	0.2500	19.2525	0.0942	0.1179
$\lambda = 0.59$	0.9000	0.2495	12.6197	0.0610	0.0715
	0.9500	0.2499	14.6120	0.0719	0.0864
	0.9750	0.2500	16.5052	0.0856	0.1023
	0.9900	0.2500	19.4287	0.1027	0.1226
$\lambda = 0.63$	0.9000	0.2495	12.4661	0.0627	0.0739
	0.9500	0.2499	14.4915	0.0745	0.0898
	0.9750	0.2500	16.5177	0.0891	0.1064
	0.9900	0.2500	19.1182	0.1066	0.1278

Table 22 Empirical Percentiles (Model 4)

λ	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
$\lambda = 0.25$	0.9000	0.2495	14.8190	0.0475	0.0724
	0.9500	0.2499	17.1342	0.0569	0.0896
	0.9750	0.2500	19.0986	0.0665	0.1057
	0.9900	0.2500	21.9067	0.0788	0.1295
$\lambda = 0.33$	0.9000	0.2495	15.2833	0.0434	0.0624
	0.9500	0.2499	17.4615	0.0510	0.0757
	0.9750	0.2500	19.5424	0.0585	0.0876
	0.9900	0.2500	22.3166	0.0692	0.1062
$\lambda = 0.41$	0.9000	0.2496	15.5408	0.0410	0.0551
	0.9500	0.2499	17.8259	0.0474	0.0656
	0.9750	0.2500	19.6557	0.0538	0.0765
	0.9900	0.2500	22.3313	0.0624	0.0899
$\lambda = 0.49$	0.9000	0.2496	15.5549	0.0407	0.0528
	0.9500	0.2499	17.6852	0.0472	0.0616
	0.9750	0.2500	19.7133	0.0539	0.0710
	0.9900	0.2500	22.4518	0.0631	0.0822
$\lambda = 0.59$	0.9000	0.2497	15.2923	0.0447	0.0557
	0.9500	0.2499	17.5324	0.0525	0.0648
	0.9750	0.2500	19.5397	0.0605	0.0760
	0.9900	0.2500	22.3962	0.0720	0.0903
$\lambda = 0.63$	0.9000	0.2497	15.0752	0.0476	0.0589
	0.9500	0.2499	17.3547	0.0559	0.0696
	0.9750	0.2500	19.4354	0.0646	0.0814
	0.9900	0.2500	22.0905	0.0795	0.0960

1. These tables are obtained by GAUSS from 10000 iteration for univariate series under known changing point.
2. Model 1: pure level shift (p=0)
   Model 2: partial level shift (p=1)
   Model 3: pure level/trend shift under continuity (p=1)
   Model 4: pure level/trend shift unrestricted (p=1)

Table 23 Empirical Percentiles (Model 1)

λ	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
$\lambda = 0.25$	0.8000	0.5586	18.2713	0.1350	0.2875
	0.8500	0.5932	19.4805	0.1462	0.3235
	0.9000	0.6489	21.0771	0.1617	0.3665
	0.9500	0.7597	23.5902	0.1884	0.4490
	0.9750	0.8889	25.8806	0.2111	0.5262
	0.9900	1.0860	28.8238	0.2489	0.6289
$\lambda = 0.33$	0.8000	0.5566	18.3554	0.1312	0.2513
	0.8500	0.5883	19.6290	0.1415	0.2780
	0.9000	0.6389	21.1676	0.1565	0.3145
	0.9500	0.7435	23.8834	0.1796	0.3732
	0.9750	0.8582	26.0735	0.2050	0.4322
	0.9900	1.0354	28.9779	0.2386	0.5182
$\lambda = 0.41$	0.8000	0.5561	18.3666	0.1316	0.2312
	0.8500	0.5870	19.5517	0.1422	0.2517
	0.9000	0.6346	21.0995	0.1570	0.2808
	0.9500	0.7261	23.8045	0.1829	0.3308
	0.9750	0.8361	26.0112	0.2066	0.3746
	0.9900	1.0312	28.9429	0.2417	0.4410
$\lambda = 0.49$	0.8000	0.5599	18.0949	0.1370	0.2211
	0.8500	0.5910	19.2929	0.1482	0.2397
	0.9000	0.6398	20.9688	0.1648	0.2684
	0.9500	0.7248	23.6388	0.1942	0.3143
	0.9750	0.8376	25.7439	0.2186	0.3538
	0.9900	1.0007	28.6544	0.2554	0.4118
$\lambda = 0.59$	0.8000	0.5656	17.7508	0.1500	0.2277
	0.8500	0.5985	18.9456	0.1642	0.2487
	0.9000	0.6510	20.5707	0.1833	0.2796
	0.9500	0.7526	23.0973	0.2167	0.3332
	0.9750	0.8620	25.2478	0.2486	0.3857
	0.9900	1.0275	27.8951	0.2940	0.4450
$\lambda = 0.63$	0.8000	0.5708	17.5077	0.1571	0.2374
	0.8500	0.6050	18.7908	0.1717	0.2615
	0.9000	0.6611	20.2266	0.1933	0.2931
	0.9500	0.7716	22.9469	0.2294	0.3529
	0.9750	0.8780	25.1282	0.2637	0.4070
	0.9900	1.0525	27.8680	0.3133	0.4828

Table 24 Empirical Percentiles (Model 2)

λ	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
$\lambda = 0.25$	0.8000	0.5418	23.2017	0.0891	0.1278
	0.8500	0.5671	24.4456	0.0951	0.1372
	0.9000	0.6041	26.2197	0.1031	0.1491
	0.9500	0.6750	29.0789	0.1155	0.1703
	0.9750	0.7527	31.3315	0.1278	0.1917
	0.9900	0.8678	33.8508	0.1459	0.2201
$\lambda = 0.33$	0.8000	0.2481	10.0997	0.0502	0.0705
	0.8500	0.2489	11.0314	0.0559	0.0780
	0.9000	0.2495	12.2511	0.0625	0.0888
	0.9500	0.2499	14.3290	0.0753	0.1071
	0.9750	0.2500	16.3372	0.0884	0.1273
	0.9900	0.2500	18.7455	0.1092	0.1506
$\lambda = 0.41$	0.8000	0.5361	21.9378	0.1098	0.1455
	0.8500	0.5554	23.2579	0.1188	0.1584
	0.9000	0.5850	25.0147	0.1329	0.1762
	0.9500	0.6445	27.7221	0.1556	0.2083
	0.9750	0.7123	30.0386	0.1806	0.2394
	0.9900	0.8096	33.2198	0.2117	0.2743
$\lambda = 0.49$	0.8000	0.5201	21.5139	0.1091	0.1533
	0.8500	0.5380	22.8187	0.1181	0.1681
	0.9000	0.5700	24.5500	0.1319	0.1899
	0.9500	0.6317	27.5603	0.1546	0.2260
	0.9750	0.6865	30.0332	0.1775	0.2580
	0.9900	0.7807	33.2174	0.2068	0.3072
$\lambda = 0.59$	0.8000	0.5336	22.7554	0.0919	0.1462
	0.8500	0.5542	24.0740	0.0981	0.1586
	0.9000	0.5902	25.9043	0.1064	0.1764
	0.9500	0.6562	28.5789	0.1208	0.2058
	0.9750	0.7262	31.0379	0.1329	0.2364
	0.9900	0.8545	33.7955	0.1507	0.2751
$\lambda = 0.63$	0.8000	0.5399	23.0548	0.0897	0.1398
	0.8500	0.5632	24.3656	0.0961	0.1507
	0.9000	0.5983	26.1481	0.1035	0.1655
	0.9500	0.6666	28.8017	0.1172	0.1912
	0.9750	0.7386	31.0472	0.1292	0.2165
	0.9900	0.8458	33.9604	0.1456	0.2512

Table 25 Empirical Percentiles (Model 3)

λ	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
$\lambda = 0.25$	0.8000	0.5450	23.2880	0.0902	0.1243
	0.8500	0.5685	24.5385	0.0961	0.1342
	0.9000	0.6051	26.3737	0.1051	0.1486
	0.9500	0.6794	29.0122	0.1192	0.1708
	0.9750	0.7663	31.2075	0.1330	0.1960
	0.9900	0.8871	34.9753	0.1510	0.2279
$\lambda = 0.33$	0.8000	0.5428	23.4794	0.0881	0.1143
	0.8500	0.5669	24.7840	0.0942	0.1229
	0.9000	0.6022	26.5431	0.1020	0.1355
	0.9500	0.6729	29.1611	0.1138	0.1554
	0.9750	0.7517	31.6765	0.1285	0.1752
	0.9900	0.8704	34.6219	0.1450	0.2031
$\lambda = 0.41$	0.8000	0.5425	23.5572	0.0875	0.1081
	0.8500	0.5660	24.9123	0.0932	0.1154
	0.9000	0.6017	26.6049	0.1011	0.1270
	0.9500	0.6704	29.3589	0.1133	0.1453
	0.9750	0.7445	31.6091	0.1263	0.1624
	0.9900	0.8506	34.4239	0.1441	0.1878
$\lambda = 0.49$	0.8000	0.5441	23.4771	0.0883	0.1055
	0.8500	0.5660	24.8415	0.0943	0.1128
	0.9000	0.6012	26.5665	0.1023	0.1239
	0.9500	0.6716	29.2091	0.1154	0.1413
	0.9750	0.7461	31.5173	0.1293	0.1569
	0.9900	0.8546	34.3038	0.1459	0.1807
$\lambda = 0.59$	0.8000	0.5472	23.2517	0.0922	0.1077
	0.8500	0.5715	24.5672	0.0988	0.1161
	0.9000	0.6062	26.2899	0.1069	0.1266
	0.9500	0.6803	28.8406	0.1224	0.1444
	0.9750	0.7546	31.2180	0.1370	0.1633
	0.9900	0.8693	34.3490	0.1564	0.1902
$\lambda = 0.63$	0.8000	0.5492	23.0769	0.0943	0.1104
	0.8500	0.5745	24.3830	0.1009	0.1189
	0.9000	0.6097	26.1462	0.1102	0.1301
	0.9500	0.6830	28.7342	0.1260	0.1500
	0.9750	0.7593	31.0806	0.1421	0.1689
	0.9900	0.8878	34.0951	0.1619	0.1970

Table 26 Empirical Percentiles (Model 4)

λ	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
$\lambda = 0.25$	0.8000	0.5349	27.1668	0.0730	0.1073
	0.8500	0.5555	28.5429	0.0777	0.1161
	0.9000	0.5874	30.4630	0.0843	0.1278
	0.9500	0.6491	33.2468	0.0953	0.1460
	0.9750	0.7173	36.1113	0.1052	0.1651
	0.9900	0.8334	39.3280	0.1204	0.1938
$\lambda = 0.33$	0.8000	0.5328	28.2005	0.0673	0.0942
	0.8500	0.5525	29.5901	0.0714	0.1011
	0.9000	0.5797	31.4751	0.0770	0.1105
	0.9500	0.6352	34.5513	0.0866	0.1256
	0.9750	0.6983	36.9969	0.0958	0.1400
	0.9900	0.7914	40.1436	0.1059	0.1608
$\lambda = 0.41$	0.8000	0.5332	28.6150	0.0645	0.0859
	0.8500	0.5500	30.1315	0.0686	0.0920
	0.9000	0.5787	31.9422	0.0738	0.0996
	0.9500	0.6307	34.8298	0.0820	0.1118
	0.9750	0.6865	37.7532	0.0892	0.1245
	0.9900	0.7653	40.8208	0.0991	0.1413
$\lambda = 0.49$	0.8000	0.5363	28.6430	0.0652	0.0833
	0.8500	0.5544	30.1026	0.0691	0.0879
	0.9000	0.5833	31.9522	0.0742	0.0945
	0.9500	0.6377	35.0183	0.0825	0.1058
	0.9750	0.6931	37.5218	0.0899	0.1172
	0.9900	0.7792	40.8163	0.1012	0.1314
$\lambda = 0.59$	0.8000	0.5447	28.0230	0.0700	0.0865
	0.8500	0.5637	29.4415	0.0747	0.0921
	0.9000	0.5931	31.2400	0.0804	0.0995
	0.9500	0.6469	34.1896	0.0911	0.1119
	0.9750	0.7052	36.8433	0.1014	0.1246
	0.9900	0.8111	39.8142	0.1140	0.1406
$\lambda = 0.63$	0.8000	0.5478	27.5203	0.0730	0.0895
	0.8500	0.5690	28.9521	0.0779	0.0958
	0.9000	0.5990	30.7865	0.0843	0.1041
	0.9500	0.6565	33.7400	0.0958	0.1182
	0.9750	0.7155	36.1956	0.1069	0.1307
	0.9900	0.8186	39.3000	0.1227	0.1502

1. These tables are obtained by GAUSS from 10000 iteration
   for bivariate series under known changing point.

Table 27 Empirical Percentiles for Sup Tests (Model 1)

n	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
n = 1	0.800	0.2500	10.2701	0.1195	0.2938
	0.850	0.2500	11.2674	0.1356	0.3289
	0.900	0.2500	12.4935	0.1588	0.3770
	0.950	0.2500	14.4704	0.1987	0.4646
	0.975	0.2500	16.3858	0.2424	0.5526
	0.990	0.2500	18.8052	0.2982	0.6690
n = 2	0.800	0.7005	21.8698	0.2196	0.4813
	0.850	0.7550	23.1178	0.2409	0.5253
	0.900	0.8375	24.6907	0.2711	0.5873
	0.950	0.9978	27.2272	0.3215	0.6891
	0.975	1.1763	29.6819	0.3740	0.7892
	0.990	1.4634	32.5821	0.4398	0.9181
n = 3	0.800	1.2970	36.7900	0.3134	0.6471
	0.850	1.4014	38.3025	0.3393	0.6985
	0.900	1.5492	40.2897	0.3742	0.7661
	0.950	1.8291	43.2363	0.4312	0.8802
	0.975	2.1340	46.0330	0.4855	0.9887
	0.990	2.5669	49.1847	0.5595	1.1346
n = 4	0.800	2.0233	55.3425	0.4054	0.8067
	0.850	2.1717	57.1543	0.4345	0.8632
	0.900	2.3851	59.5519	0.4738	0.9400
	0.950	2.7735	63.1246	0.5357	1.0625
	0.975	3.1798	66.3569	0.5938	1.1768
	0.990	3.7945	70.1290	0.6742	1.3162
n = 5	0.800	2.8821	77.5691	0.4954	0.9619
	0.850	3.0875	79.5690	0.5276	1.0230
	0.900	3.3765	82.2933	0.5701	1.1034
	0.950	3.8949	86.2202	0.6384	1.2404
	0.975	4.4437	89.7318	0.6994	1.3682
	0.990	5.1828	94.3984	0.7837	1.5153

Table 28 Empirical Percentiles for Sup Tests (Model 2)

n	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
n = 1	0.800	0.2500	14.5149	0.0712	0.1159
	0.850	0.2500	15.5144	0.0794	0.1275
	0.900	0.2500	16.9126	0.0909	0.1436
	0.950	0.2500	19.1359	0.1107	0.1716
	0.975	0.2500	21.0785	0.1316	0.2001
	0.990	0.2500	23.6560	0.1575	0.2391
n = 2	0.800	0.6632	27.8936	0.1322	0.1956
	0.850	0.7019	29.2155	0.1427	0.2102
	0.900	0.7589	30.8900	0.1573	0.2298
	0.950	0.8670	33.5073	0.1818	0.2639
	0.975	0.9857	35.8807	0.2059	0.2958
	0.990	1.1709	38.7996	0.2377	0.3388
n = 3	0.800	1.1553	44.8072	0.1869	0.2696
	0.850	1.2223	46.3811	0.1987	0.2863
	0.900	1.3129	48.4144	0.2156	0.3085
	0.950	1.4773	51.6770	0.2432	0.3455
	0.975	1.6463	54.5468	0.2698	0.3835
	0.990	1.8884	58.0719	0.3061	0.4302
n = 4	0.800	1.7175	65.2120	0.2394	0.3412
	0.850	1.8071	67.0840	0.2523	0.3596
	0.900	1.9336	69.3687	0.2707	0.3846
	0.950	2.1577	73.1376	0.3004	0.4251
	0.975	2.3849	76.4737	0.3290	0.4643
	0.990	2.6885	80.2510	0.3667	0.5190
n = 5	0.800	2.3551	89.1968	0.2924	0.4097
	0.850	2.4744	91.4069	0.3066	0.4299
	0.900	2.6351	94.0622	0.3262	0.4572
	0.950	2.9252	98.4637	0.3575	0.5018
	0.975	3.2159	102.2493	0.3873	0.5430
	0.990	3.5917	107.0198	0.4267	0.5963

Table 29 Empirical Percentiles for Sup Tests (Model 3)

n	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
n = 1	0.800	0.2500	12.6307	0.0645	0.0900
	0.850	0.2500	13.6553	0.0713	0.1002
	0.900	0.2500	14.9880	0.0809	0.1145
	0.950	0.2500	17.1639	0.0970	0.1386
	0.975	0.2500	19.3133	0.1141	0.1637
	0.990	0.2500	22.0483	0.1368	0.1960
n = 2	0.800	0.6091	26.4891	0.1181	0.1608
	0.850	0.6424	27.7868	0.1269	0.1736
	0.900	0.6932	29.5474	0.1391	0.1912
	0.950	0.7937	32.4234	0.1592	0.2199
	0.975	0.9028	35.0448	0.1789	0.2490
	0.990	1.0736	38.0954	0.2044	0.2878
n = 3	0.800	1.0699	43.4910	0.1693	0.2278
	0.850	1.1315	45.1050	0.1793	0.2425
	0.900	1.2196	47.2527	0.1938	0.2626
	0.950	1.3799	50.5990	0.2173	0.2951
	0.975	1.5589	53.6760	0.2386	0.3268
	0.990	1.7992	57.3629	0.2692	0.3687
n = 4	0.800	1.6118	63.9991	0.2196	0.2929
	0.850	1.7002	65.9334	0.2314	0.3093
	0.900	1.8287	68.4353	0.2473	0.3310
	0.950	2.0475	72.4165	0.2721	0.3662
	0.975	2.2823	76.0654	0.2951	0.4009
	0.990	2.6039	80.2543	0.3229	0.4425
n = 5	0.800	2.2323	88.1603	0.2691	0.3556
	0.850	2.3514	90.3943	0.2818	0.3732
	0.900	2.5217	93.3097	0.2977	0.3970
	0.950	2.8129	97.7634	0.3255	0.4363
	0.975	3.1130	101.7885	0.3513	0.4735
	0.990	3.5512	106.6342	0.3849	0.5203

Table 30 Empirical Percentiles for Sup Tests (Model 4)

n	Percentile	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
n = 1	0.800	0.2500	16.9971	0.0587	0.0932
	0.850	0.2500	18.0852	0.0647	0.1023
	0.900	0.2500	19.6210	0.0728	0.1156
	0.950	0.2500	21.9213	0.0872	0.1382
	0.975	0.2500	24.0336	0.1027	0.1602
	0.990	0.2500	26.8095	0.1224	0.1893
n = 2	0.800	0.7046	31.5807	0.1388	0.1589
	0.850	0.7521	32.9476	0.1488	0.1706
	0.900	0.8224	34.7221	0.1625	0.1868
	0.950	0.9572	37.5606	0.1840	0.2135
	0.975	1.0991	40.0479	0.2032	0.2401
	0.990	1.3269	43.0335	0.2290	0.2718
n = 3	0.800	1.2506	50.6911	0.1966	0.2204
	0.850	1.3324	52.2817	0.2084	0.2339
	0.900	1.4511	54.4116	0.2234	0.2514
	0.950	1.6736	57.6043	0.2479	0.2816
	0.975	1.8979	60.5734	0.2689	0.3106
	0.990	2.2685	64.0251	0.2953	0.3487
n = 4	0.800	1.8869	72.8866	0.2515	0.2800
	0.850	2.0066	74.8994	0.2636	0.2951
	0.900	2.1806	77.4186	0.2790	0.3158
	0.950	2.4818	81.3086	0.3048	0.3487
	0.975	2.8055	84.9986	0.3271	0.3778
	0.990	3.2935	89.4487	0.3575	0.4182
n = 5	0.800	2.6328	98.9963	0.3054	0.3401
	0.850	2.7935	101.2260	0.3182	0.3562
	0.900	3.0171	104.0339	0.3350	0.3782
	0.950	3.4283	108.3464	0.3611	0.4128
	0.975	3.8511	112.2954	0.3866	0.4442
	0.990	4.4267	117.1247	0.4144	0.4862

1. Percentiles are obtained by GAUSS/Fortran from 50000 iteration where $\lambda \in (0.15, 0.85)$ and interval 0.02.

2. Model 1: pure level shift (p=0)
   Model 2: partial level shift (p=1)
   Model 3: pure level/trend shift under continuity (p=1)
   Model 4: pure level/trend shift unrestricted (p=1)

Table 31 Empirical Power of Sup Tests

$$DGP\,1: x_t = \begin{bmatrix} 1.0 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, e_t \sim iidN(0,\Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

M	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
	Part(a) Univariate Tests							
	univariate tests for x_{1t}				univariate tests for x_{2t}			
	T = 100							
1	0.34	0.00	0.93	0.91	0.29	0.00	0.89	0.85
2	0.14	0.00	0.83	0.79	0.10	0.00	0.77	0.72
3	0.00	0.00	0.77	0.72	0.00	0.00	0.70	0.65
4	0.00	0.00	0.77	0.78	0.00	0.00	0.69	0.69
	T = 200							
1	0.67	0.02	0.98	0.97	0.64	0.02	0.97	0.95
2	0.60	0.00	0.93	0.89	0.55	0.00	0.87	0.80
3	0.31	0.00	0.92	0.87	0.27	0.00	0.86	0.76
4	0.23	0.00	0.90	0.87	0.19	0.00	0.81	0.76
	T = 400							
1	0.89	0.25	1.00	1.00	0.88	0.24	0.99	0.99
2	0.87	0.08	0.99	0.98	0.08	0.07	0.95	0.93
3	0.69	0.03	0.98	0.97	0.66	0.02	0.95	0.92
4	0.60	0.01	0.98	0.97	0.55	0.00	0.95	0.91
	Part(b) Univariate and Multivariate Tests							
	univariate tests				multivariate tests			
	T = 100							
1	0.38	0.00	0.96	0.94	0.88	0.19	0.93	0.90
2	0.17	0.00	0.92	0.89	0.65	0.04	0.85	0.79
3	0.00	0.00	0.87	0.84	0.59	0.05	0.79	0.75
4	0.00	0.00	0.89	0.89	0.44	0.02	0.70	0.80
	T = 200							
1	0.71	0.02	0.99	0.98	0.97	0.62	0.98	0.97
2	0.65	0.00	0.95	0.92	0.90	0.35	0.93	0.91
3	0.35	0.00	0.94	0.91	0.85	0.33	0.91	0.88
4	0.26	0.00	0.94	0.91	0.82	0.29	0.89	0.90
	T = 400							
1	0.91	0.27	1.00	1.00	1.00	0.90	1.00	1.00
2	0.89	0.09	0.99	0.98	0.98	0.78	0.98	0.97
3	0.72	0.03	0.99	0.98	0.97	0.65	0.98	0.97
4	0.63	0.01	0.99	0.98	0.97	0.72	0.98	0.97

Table 32 Empirical Power of Sup Tests

$$DGP\, 2: x_t = \begin{bmatrix} 1.0 & 0.2 \\ 0.0 & 0.8 \end{bmatrix} x_{t-1} + e_t, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

M	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
	Part(a) Univariate Tests							
	univariate tests for x_{1t}				univariate tests for x_{2t}			
	T = 100							
1	0.42	0.00	0.96	0.95	0.01	0.00	0.28	0.20
2	0.22	0.00	0.91	0.88	0.00	0.00	0.34	0.28
3	0.01	0.00	0.84	0.82	0.00	0.00	0.30	0.23
4	0.00	0.00	0.86	0.89	0.00	0.00	0.36	0.30
	T = 200							
1	0.71	0.03	0.99	0.99	0.00	0.00	0.07	0.06
2	0.68	0.00	0.97	0.95	0.00	0.00	0.13	0.10
3	0.38	0.00	0.94	0.91	0.00	0.00	0.11	0.11
4	0.28	0.00	0.96	0.94	0.00	0.00	0.14	0.14
	T = 400							
1	0.89	0.27	1.00	1.00	0.00	0.00	0.02	0.07
2	0.90	0.10	0.99	0.99	0.00	0.00	0.06	0.06
3	0.73	0.04	0.99	0.99	0.00	0.00	0.04	0.07
4	0.64	0.01	0.99	0.99	0.00	0.00	0.06	0.08
	Part(b) Univariate and Multivariate Tests							
	univariate tests				multivariate tests			
	T = 100							
1	0.43	0.00	0.96	0.96	0.94	0.52	0.94	0.92
2	0.23	0.00	0.93	0.91	0.82	0.20	0.89	0.82
3	0.01	0.00	0.88	0.85	0.75	0.30	0.83	0.72
4	0.00	0.00	0.91	0.91	0.66	0.16	0.75	0.82
	T = 200							
1	0.71	0.03	0.99	0.99	0.97	0.91	0.96	0.95
2	0.68	0.00	0.97	0.95	0.89	0.74	0.89	0.85
3	0.38	0.00	0.94	0.92	0.87	0.76	0.85	0.79
4	0.28	0.00	0.96	0.95	0.85	0.67	0.84	0.84
	T = 400							
1	0.89	0.27	1.00	1.00	1.00	0.99	0.99	0.99
2	0.90	0.10	0.99	0.99	0.98	0.96	0.97	0.96
3	0.73	0.04	0.99	0.99	0.97	0.96	0.96	0.94
4	0.64	0.01	0.99	0.99	0.97	0.96	0.96	0.95

Table 33 Empirical Size of Sup Tests

$$DGP\ 3: x_t = \begin{bmatrix} 0.8 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

M	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$	LM_I	LM_{II}	$SBDH_I$	$SBDH_{II}$
	Part(a) Univariate Tests							
	univariate tests for x_{1t}				univariate tests for x_{2t}			
	T = 100							
1	0.01	0.00	0.27	0.18	0.03	0.00	0.45	0.35
2	0.00	0.00	0.34	0.28	0.01	0.00	0.47	0.40
3	0.00	0.00	0.30	0.25	0.00	0.00	0.40	0.32
4	0.00	0.00	0.35	0.30	0.00	0.00	0.48	0.42
	T = 200							
1	0.00	0.00	0.08	0.06	0.02	0.00	0.20	0.12
2	0.00	0.00	0.13	0.10	0.02	0.00	0.19	0.14
3	0.00	0.00	0.09	0.10	0.01	0.00	0.19	0.13
4	0.00	0.00	0.14	0.13	0.00	0.00	0.20	0.16
	T = 400							
1	0.00	0.00	0.03	0.06	0.00	0.00	0.03	0.05
2	0.00	0.00	0.07	0.08	0.00	0.00	0.05	0.05
3	0.00	0.00	0.04	0.09	0.00	0.00	0.03	0.06
4	0.00	0.00	0.06	0.09	0.00	0.00	0.04	0.07
	Part(b) Univariate and Multivariate Tests							
	univariate tests				multivariate tests			
	T = 100							
1	0.03	0.00	0.54	0.44	0.38	0.05	0.44	0.32
2	0.01	0.00	0.62	0.54	0.28	0.01	0.47	0.38
3	0.00	0.00	0.54	0.46	0.26	0.01	0.41	0.36
4	0.00	0.00	0.64	0.57	0.19	0.01	0.34	0.45
	T = 200							
1	0.02	0.00	0.24	0.17	0.23	0.04	0.19	0.13
2	0.02	0.00	0.27	0.21	0.20	0.02	0.23	0.19
3	0.01	0.00	0.24	0.20	0.21	0.01	0.22	0.21
4	0.00	0.00	0.30	0.25	0.17	0.01	0.14	0.26
	T = 400							
1	0.00	0.00	0.05	0.09	0.12	0.01	0.04	0.09
2	0.00	0.00	0.10	0.11	0.11	0.01	0.10	0.12
3	0.00	0.00	0.06	0.12	0.13	0.01	0.08	0.15
4	0.00	0.00	0.09	0.13	0.08	0.01	0.02	0.17

1. Size and power are obtained by GAUSS from 2000 iteration.
2. Quadratic kernel is used for longrun variance estimation.
3. $\lambda \in (0.15, 0.85)$ at interval 0.02.

Table 34 Empirical Size of LM_I, LM_{II} and $SBDH$

$$x_t = \begin{bmatrix} 0.8 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

		Univariate tests		Multivariate tests	
		s^*	l_2	s^*	l_2
		a. Standard			
T=100	LM_I	0.21	0.09	0.56	0.07
	LM_{II}	0.13	0.01	0.06	0.00
	$SBDH$	0.21	0.07	0.50	0.05
T=200	LM_I	0.09	0.11	0.27	0.11
	LM_{II}	0.03	0.03	0.03	0.00
	$SBDH$	0.09	0.10	0.24	0.10
T=400	LM_I	0.08	0.10	0.04	0.11
	LM_{II}	0.05	0.05	0.00	0.04
	$SBDH$	0.08	0.10	0.04	0.11
		b. Demeaned			
T=100	LM_I	0.17	0.07	0.24	0.07
	LM_{II}	0.00	0.00	0.06	0.00
	$SBDH_T$	0.36	0.03	0.24	0.03
	$SBDH_B$	0.34	0.09	0.25	0.08
T=200	LM_I	0.11	0.07	0.10	0.11
	LM_{II}	0.01	0.00	0.01	0.00
	$SBDH_T$	0.15	0.09	0.07	0.08
	$SBDH_B$	0.16	0.12	0.10	0.11
T=400	LM_I	0.09	0.09	0.09	0.13
	LM_{II}	0.02	0.02	0.01	0.02
	$SBDH_T$	0.08	0.11	0.07	0.10
	$SBDH_B$	0.10	0.12	0.10	0.13
		c. Demeaned and detrended			
T=100	LM_I	0.06	0.06	0.17	0.03
	LM_{II}	0.00	0.00	0.02	0.00
	$SBDH_T$	0.31	0.03	0.27	0.07
	$SBDH_B$	0.30	0.07	0.27	0.13
T=200	LM_I	0.08	0.07	0.10	0.09
	LM_{II}	0.00	0.00	0.01	0.00
	$SBDH_T$	0.13	0.06	0.09	0.08
	$SBDH_B$	0.14	0.10	0.13	0.12
T=400	LM_I	0.08	0.08	0.08	0.12
	LM_{II}	0.00	0.00	0.00	0.00
	$SBDH_T$	0.08	0.11	0.08	0.11
	$SBDH_B$	0.10	0.13	0.11	0.14

Table 35 Empirical Power of LM_I, LM_{II} and $SBDH$

$$x_t = \begin{bmatrix} 1.0 & 0.0 \\ 0.2 & 0.8 \end{bmatrix} x_{t-1} + e_t, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

		Univariate tests		Multivariate tests	
		s^*	l_2	s^*	l_2
		a. Standard			
T=100	LM_I	0.88	0.60	0.75	0.51
	LM_{II}	0.83	0.10	0.31	0.00
	$SBDH$	0.86	0.20	0.66	0.17
T=200	LM_I	0.91	0.74	0.87	0.75
	LM_{II}	0.86	0.62	0.48	0.27
	$SBDH$	0.95	0.53	0.84	0.52
T=400	LM_I	0.94	0.81	0.95	0.84
	LM_{II}	0.88	0.73	0.60	0.79
	$SBDH$	0.99	0.71	0.94	0.71
		b. Demeaned			
T=100	LM_I	0.68	0.02	0.87	0.14
	LM_{II}	0.03	0.00	0.23	0.00
	$SBDH_T$	0.92	0.30	0.92	0.24
	$SBDH_B$	0.91	0.56	0.91	0.54
T=200	LM_I	0.78	0.02	0.95	0.31
	LM_{II}	0.34	0.00	0.61	0.00
	$SBDH_T$	0.98	0.60	0.98	0.60
	$SBDH_B$	0.97	0.70	0.97	0.71
T=400	LM_I	0.85	0.41	0.98	0.65
	LM_{II}	0.53	0.00	0.83	0.04
	$SBDH_T$	1.00	0.79	1.00	0.79
	$SBDH_B$	1.00	0.83	1.00	0.82
		c. Demeaned and detrended			
T=100	LM_I	0.14	0.03	0.56	0.05
	LM_{II}	0.00	0.00	0.11	0.00
	$SBDH_T$	0.75	0.12	0.79	0.19
	$SBDH_B$	0.73	0.33	0.78	0.40
T=200	LM_I	0.45	0.03	0.82	0.21
	LM_{II}	0.00	0.00	0.33	0.00
	$SBDH_T$	0.90	0.41	0.91	0.44
	$SBDH_B$	0.88	0.57	0.90	0.61
T=400	LM_I	0.63	0.03	0.95	0.36
	LM_{II}	0.12	0.00	0.61	0.01
	$SBDH_T$	0.98	0.74	0.98	0.73
	$SBDH_B$	0.97	0.82	0.98	0.82

Table 36 Empirical Power of LM_I, LM_{II} and $SBDH$

$$x_t = \begin{bmatrix} 1.0 & 0.2 \\ 0.0 & 0.8 \end{bmatrix} x_{t-1} + e_t, e_t \sim iidN(0, \Omega), \Omega = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}.$$

		Univariate tests		Multivariate tests	
		s^*	l_2	s^*	l_2
		a. Standard			
T=100	LM_I	0.86	0.57	0.78	0.50
	LM_{II}	0.81	0.09	0.21	0.00
	$SBDH$	0.84	0.20	0.65	0.16
T=200	LM_I	0.90	0.73	0.91	0.76
	LM_{II}	0.85	0.62	0.36	0.57
	$SBDH$	0.94	0.52	0.84	0.48
T=400	LM_I	0.93	0.80	0.98	0.87
	LM_{II}	0.88	0.73	0.43	0.95
	$SBDH$	0.99	0.71	0.94	0.68
		b. Demeaned			
T=100	LM_I	0.69	0.05	0.87	0.28
	LM_{II}	0.03	0.00	0.52	0.00
	$SBDH_T$	0.91	0.28	0.85	0.27
	$SBDH_B$	0.91	0.57	0.84	0.56
T=200	LM_I	0.79	0.06	0.95	0.54
	LM_{II}	0.34	0.00	0.88	0.01
	$SBDH_T$	0.98	0.60	0.95	0.62
	$SBDH_B$	0.98	0.72	0.95	0.73
T=400	LM_I	0.85	0.44	0.99	0.88
	LM_{II}	0.54	0.01	0.98	0.20
	$SBDH_T$	1.00	0.80	0.99	0.80
	$SBDH_B$	1.00	0.82	0.99	0.83
		c. Demeaned and detrended			
T=100	LM_I	0.19	0.05	0.61	0.12
	LM_{II}	0.00	0.00	0.31	0.00
	$SBDH_T$	0.73	0.12	0.65	0.24
	$SBDH_B$	0.71	0.35	0.65	0.46
T=200	LM_I	0.46	0.05	0.82	0.49
	LM_{II}	0.00	0.00	0.72	0.00
	$SBDH_T$	0.92	0.42	0.83	0.48
	$SBDH_B$	0.90	0.58	0.81	0.62
T=400	LM_I	0.64	0.06	0.96	0.75
	LM_{II}	0.13	0.00	0.94	0.11
	$SBDH_T$	0.99	0.73	0.96	0.73
	$SBDH_B$	0.99	0.81	0.95	0.81

Table 37 Test results fot the Kugler-Neusser data
a. Univariate tests for the null of level-stationarity

	LM_I	LM_{II}	$SBDH_T$	$SBDH_B$
USA	0.3900	0.1617	0.2453	0.4333
Japan	0.1804	2.2394	0.0816	0.1317
UK	0.0943	0.8829	0.1095	0.2574
FRG	0.1872	2.7845	0.0682	0.1170
France	0.2497	3.3193	0.0752	0.0755
Switzerland	0.1484	2.3840	0.0638	0.1472

Critical values at 5% level are 0.2496, 7.9924, 0.2477 and 0.4589 for LM_I, LM_{II}, $SBDH_T$ and $SBDH_B$, respectively.

b. Multivariate tests for the null of level-stationarity

	LM_I	LM_{II}	$SBDH_T$	$SBDH_B$
All	2.2916	29.7916	0.5682	0.9285

Critical values at 10% level are 5.0410, 88.1664, 0.8412 and 0.8412 for LM_I, LM_{II}, $SBDH_T$ and $SBDH_B$, respectively.

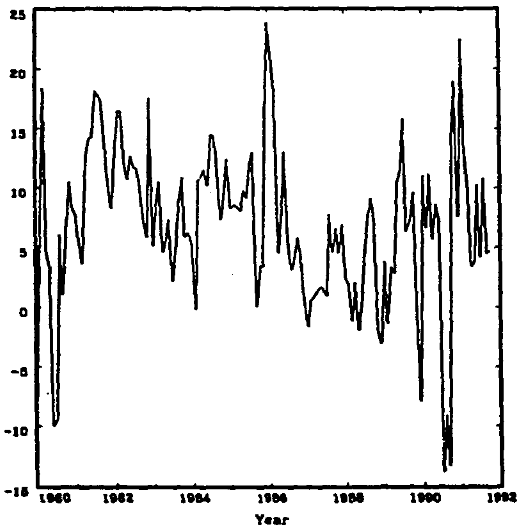

Figure 1 USA Real Interest Rate

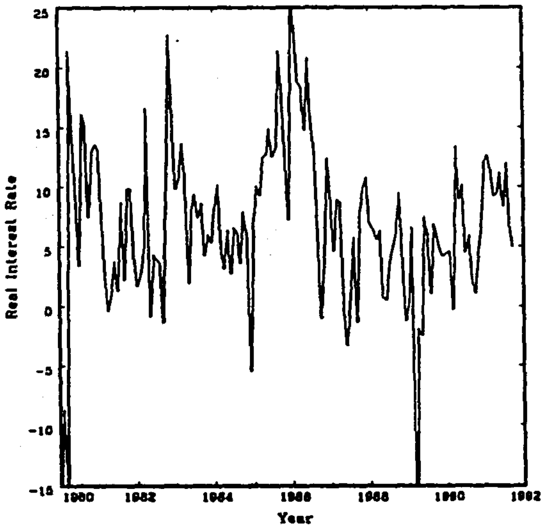

Figure 2: Japan Real Interest Rate

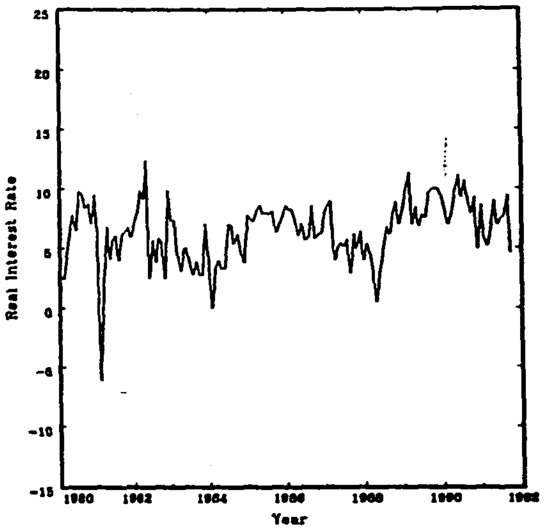

Figure 3: UK Real Interest Rate

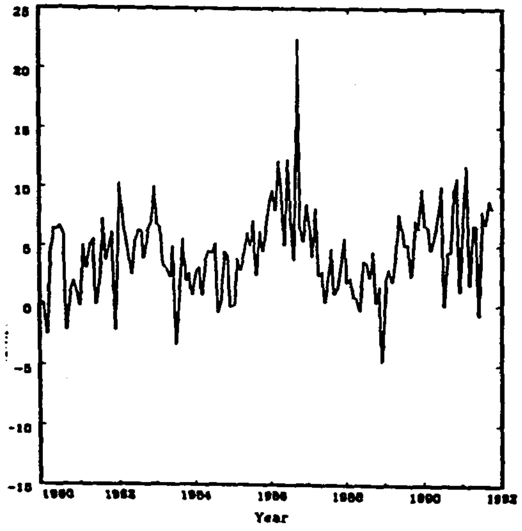

Figure 4. FRG Real Interest Rate

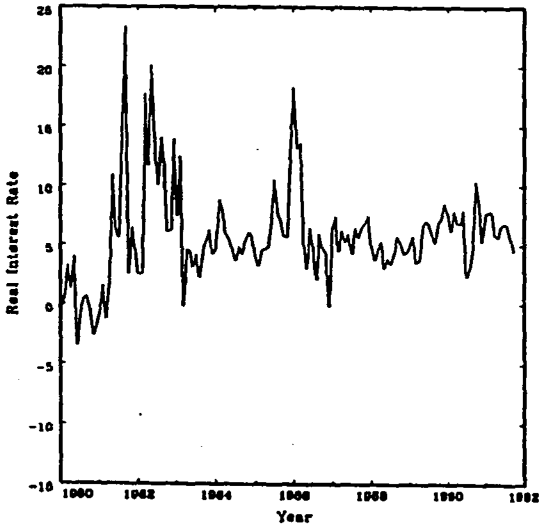

Figure 5: France Real Interest Rate

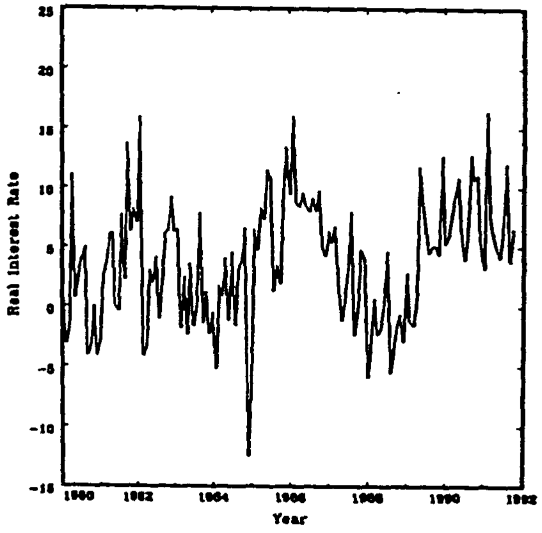

Figure 6: Switzerland Real Interest Rate

Appendix A

Proofs for Chapter II

Lemma A. *Let an $n \times n$ matrix A be partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} is a constant matrix and A_{12}, A_{21} and A_{22} are random matrices with continuous distributions. Suppose that rank$(A_{11}) = n_1$, rank$(A_{22}) = n_2$ a.s. and $A_{12}A_{22}^{-1}A_{21}$ is a random matrix. Then, the inverse of the matrix A exists a.s.*

Proof:

In the light of a formula for the partitioned inverse, the inverse of the matrix A exists if the inverse of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ exists because it is assumed that A_{22} has full rank *a.s.* By the given assumption $A_{21}A_{22}^{-1}A_{12}$ is a random matrix with a continuous distribution. Now write

$$A_{11} - A_{12}A_{22}^{-1}A_{21} = [a_1 - b_1, \cdots, a_{n_1} - b_{n_1}]. \tag{A.1}$$

Suppose that

$$\alpha_1(a_1 - b_1) + \cdots + \alpha_{n_1}(a_{n_1} - b_{n_1}) = 0 \quad a.s. \tag{A.2}$$

for some random variables $\alpha_1, \cdots, \alpha_{n_1}$. Then, we have either

$$\alpha_1 a_1 + \cdots + \alpha_{n_1} a_{n_1} = \alpha_1 b_1 + \cdots + \alpha_{n_1} b_{n_1} \neq 0 \quad a.s. \tag{A.3}$$

or

$$\alpha_1 a_1 + \cdots + \alpha_{n_1} a_{n_1} = 0 \ a.s. \text{ and } \alpha_1 b_1 + \cdots + \alpha_{n_1} b_{n_1} = 0 \ a.s. \tag{A.4}$$

But $Pr[\alpha_1 a_1 + \cdots + \alpha_{n_1} a_{n_1} = \alpha_1 b_1 + \cdots + \alpha_{n_1} b_{n_1} \neq 0] = 0$ for any random variables $\alpha_1, \cdots, \alpha_{n_1}$, because $b_1, \cdots, b_{n_1}$ have continuous distributions. Thus, (A.4) holds, which implies that $\alpha_1 = \cdots = \alpha_{n_1} = 0$ $a.s.$ by the given assumption on the rank of the matrices A_{11}. Therefore, the matrix $A_{11} - A_{12}A_{22}^{-1}A_{21}$ has full rank $a.s.$ and its inverse exists $a.s.$

Proof of Theorem 1.

(a) Because $\hat{\Omega}_l \xrightarrow{p} \Omega$ and $\hat{\Omega}_1 \xrightarrow{p} \Omega_1$, we obtain the required results by using the weak convergence results found in Phillips and Durlauf (1986) and Park and Phillips (1988).

(b) First, we consider the case where x_t is not cointegrated. We assume without loss of generality that

$$x_t^{(1)}, x_t^{(2)}, \cdots, x_t^{(s)} = I(0), \ 0 \leq s < n \tag{A.5}$$

($i.e., k_i = 0, i = 1, \cdots, s$) and $x_t^{(i)} = I(k_i)$, $s < i \leq n, k_i \geq 1$ under the alternative. By using the weak law of large numbers and the weak convergence results found in Phillips and Durlauf (1986) and Chan and Wei (1988), we have

$$\sum_{t=2}^{T} \Delta S_t^{(i)} S_{t-1}^{(j)} = O_p(T^{k_i+k_j+1}) \tag{A.6}$$

and

$$\sum_{t=2}^{T} S_{t-1}^{(i)} S_{t-1}^{(j)} = O_p(T^{k_i+k_j+2}), \tag{A.7}$$

which imply

$$D^{-1}T^{-1}\sum_{t=2}^{T} \Delta S_t S_{t-1}' D^{-1} \ = \ O_p(1), \tag{A.8}$$

$$D^{-1}T^{-2}\sum_{t=2}^{T} S_{t-1}S'_{t-1}D^{-1} \;=\; O_p(1), \tag{A.9}$$

where $D = diag[T^{k_1}, \cdots, T^{k_n}]$. Also note that

$$D^{-1}T^{-2}\sum_{t=2}^{T} S_{t-1}S'_{t-1}D^{-1} \text{ is nonsingular in the limit } a.s. \tag{A.10}$$

due to lemma 3.1.1 in Chan and Wei. Further, when $i \leq s$ and $j \leq s$, we obtain by a standard theory for spectral density estimation

$$[\hat{\Omega}_l^{(i,j)}]_{i,j=1}^{s} \rightarrow \Omega_{l11}, \tag{A.11}$$

where Ω_{l11} is a nonsingular constant matrix by given assumptions. By contrast, when either $i > s$ or $j > s$, we have as in KPSS (1992)

$$T^{-(k_i+k_j-1)}l^{-1}\hat{\Omega}_l^{(i,j)} \Rightarrow K\Omega_l^{(i,j)}\int_0^1 W_{k_i}(r)W_{k_j}(r)dr, \tag{A.12}$$

where $l = O(T^{\delta})$, $K = \int_{-1}^{1} k(x)dx$, $\Omega_l^{(i,j)}/(2\pi)$ is the cospectrum of $\Delta^{k_i}x_t^{(i)}$ and $\Delta^{k_j}x_t^{(j)}$ at the zero frequency, $W_m(r) = \int_0^r W_{m-1}(s)ds$, $W_1(r) = W(r)$ and $W_0(r) = dW(r)$. (A.11) and (A.12) imply that

$$F^{-1}D^{-1}\hat{\Omega}_l D^{-1}F^{-1} = O_p(1), \tag{A.13}$$

where $F = diag[1, \cdots, \overset{s-th}{1}, T^{(\delta-1)/2}, \cdots, T^{(\delta-1)/2}]$. Using Chan and Wei's lemma 3.1.1, we also find that $[F^{-1}D^{-1}\hat{\Omega}_l^{(i,j)}D^{-1}F^{-1}]_{i,j=s+1}^{n}$ is nonsingular in the limit $a.s.$, from which, together with (A.11), we deduce by using Lemma A that

$$F^{-1}D^{-1}\hat{\Omega}_l D^{-1}F^{-1} \text{ is nonsingular in the limit } a.s. \tag{A.14}$$

Using the same arguments as for (A.13), we readily obtain

$$F^{-1}D^{-1}\hat{\Omega}_1 D^{-1}F^{-1} = O_p(1). \tag{A.15}$$

Hence, upon writing

$$
\begin{aligned}
LM_I^m &= tr\{F^{-1}(D^{-1}T^{-1}\sum_{t=2}^{T}\Delta S_t S_{t-1}' D^{-1} - D^{-1}\hat{\Omega}_1' D^{-1}) \\
&\quad \cdot F^{-1}(F^{-1}D^{-1}\hat{\Omega}_l D^{-1}F^{-1})^{-1} \\
&\quad \cdot F^{-1}(D^{-1}T^{-1}\sum_{t=2}^{T} S_{t-1}\Delta S_t' D^{-1} - D^{-1}\hat{\Omega}_1 D^{-1}) \\
&\quad \cdot F^{-1}(F^{-1}D^{-1}\hat{\Omega}_l D^{-1}F^{-1})^{-1}\}, \qquad (A.16) \\
LM_{II}^m &= tr\{F^{-1}(D^{-1}T^{-1}\sum_{t=2}^{T}\Delta S_t S_{t-1}' D^{-1} - D^{-1}\hat{\Omega}_1' D^{-1}) \\
&\quad \cdot F^{-1}F(D^{-1}T^{-2}\sum_{t=2}^{T} S_{t-1}S_{t-1}' D^{-1})^{-1}F \\
&\quad \cdot F^{-1}(D^{-1}T^{-1}\sum_{t=2}^{T} S_{t-1}\Delta S_t' D^{-1} - D^{-1}\hat{\Omega}_1 D^{-1})F^{-1} \\
&\quad \cdot F(F^{-1}D^{-1}\hat{\Omega}_l D^{-1}F^{-1})^{-1}\}, \qquad (A.17) \\
SBDH^m &= tr\{F^{-1}(D^{-1}T^{-2}\sum_{t=1}^{T} S_t S_t' D^{-1})F^{-1}(F^{-1}D^{-1}\hat{\Omega}_l D^{-1}F^{-1})^{-1}\} (A.18)
\end{aligned}
$$

and using (A.9), (A.10), (A.13), (A.14) and (A.15), we obtain the desired results.

Next, we consider the case where the nonstationary element of x_t is cointegrated. We construct an $n \times n$ nonsingular matrix G [cf. Choi (1991)] such that $G = \begin{bmatrix} I_s & 0 \\ 0 & c \end{bmatrix}$. The matrix $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ is constructed such that the $m \times (n - s)$ matrix C_1 is a cointegrating matrix as in Section 2 and the $i - th$ row ($i = 1, \cdots, n - s - m$) of the $(n - s - m) \times (n - s)$ matrix C_2 is $[0, \cdots, 0, \overset{i-th}{1}, 0, \cdots, 0]$. Hence, letting $[x_t^{(s+1)}, \cdots, x_t^{(n)}] = z_t$, we have $C_1 z_t = [I(l_1), \cdots, I(l_m)]'$ and $C_2 z_t = [I(k_{s+1}), \cdots, I(k_{n-m})]'$. Because the test statistics are invariant with respect to the linear transformation G, we obtain the same results as in the case of non-cointegrated

x_t by redefining

$$D = diag[T^{k_1}, \cdots, T^{k_s}, T^{l_1}, \cdots, T^{l_m}, T^{k_{s+1}}, \cdots, T^{k_{n-m}}] \tag{A.19}$$

and

$$F = diag[1, \cdots, T^s, f_1, \cdots, f_m, T^{(\delta-1)/2}, \cdots, T^{(\delta-1)/2}] \tag{A.20}$$

with $f_i = 1(l_j = 0) + 1(l_j > 0)T^{(\delta-1)/2}$, $(j = 1, \cdots, m)$.

Lemma B. *Suppose that assumptions A1-A9 hold. Under the null hypothesis (2.1),*

$$(i)\ T^{-1}\sum_{t=2}^{T} \Delta\tilde{S}_t\tilde{S}_{t-1}' \Rightarrow \int_0^1 d\tilde{B}(r)\tilde{B}(r)'dr + \Omega_1, \tag{A.21}$$

$$(ii)\ T^{-2}\sum_{t=2}^{T} \tilde{S}_t\tilde{S}_t' \Rightarrow \int_0^1 \tilde{B}(r)\tilde{B}(r)'dr, \tag{A.22}$$

$$(iii)\ T^{-2}\sum_{t=2}^{T} \bar{S}_t\bar{S}_t' \Rightarrow \int_0^1 \bar{B}(r)\bar{B}(r)'dr, \tag{A.23}$$

$$(iv)\ \tilde{\Omega}_1 \rightarrow \Omega_1, \tag{A.24}$$

$$(v)\ \tilde{\Omega}_l \rightarrow \Omega_l, \tag{A.25}$$

$$(vi)\ \bar{\Omega}_l \rightarrow \Omega_l, \tag{A.26}$$

where

$$B(r) = \Omega_l^{1/2}W(r), \tag{A.27}$$

$$\bar{B}(r) = B(r) - \bar{\eta}_0 r^1/1 - \cdots - \bar{\eta}_p r^{p+1}/(p+1), \tag{A.28}$$

$$\tilde{B}(r) = B(r) - \tilde{\psi}_0 r^1/1 - \cdots - \tilde{\psi}_p r^{p+1}/(p+1), \tag{A.29}$$

$\bar{\eta}_i$ *and* $\tilde{\psi}_i$ *minimize the least squares criteria in the* L_2 *norm, respectively,*

$$\int_0^1 \|B(r) - \eta_0 r^0 - \cdots - \eta_p r^p\|^2 dr, \tag{A.30}$$

$$\int_0^1 \|B(r) - \psi_0 r^1/1 - \cdots - \psi_p r^{p+1}/(p+1)\|^2 dr. \tag{A.31}$$

Proof:

(i) Write

$$\begin{aligned}\sum_{t=2}^{T} \Delta\tilde{S}_{t-1}\tilde{S}_{t-1}' &= \sum_{t=2}^{T}\{x_t - (\tilde{\delta}_0 - \delta_0)t^0 - \cdots - (\tilde{\delta}_p - \delta_p)t^p\} \\ &\quad \cdot\{S_{t-1} - (\tilde{\delta}_0 - \delta_0)\sum_{j=1}^{t-1} j^0 - \cdots - (\tilde{\delta}_p - \delta_p)\sum_{j=1}^{t-1} j^p\}'. \end{aligned} \tag{A.32}$$

We have as in Phillips and Durlauf (1986) and Park and Phillips (1988)

$$T^{-1}\sum_{t=2}^{T} x_t S_{t-1}' \Rightarrow \int_0^1 dB(r)B(r)'dr + \Omega_1, \tag{A.33}$$

$$T^{-(3/2+n)}\sum_{t=2}^{T} t^n S_{t-1} \Rightarrow \int_0^1 r^n B(r)dr, \tag{A.34}$$

$$T^{-(3/2+n)}\sum_{t=2}^{T} x_t \sum_{j=1}^{t-1} j^n \Rightarrow \int_0^1 r^{n+1} dB(r)dr/(n+1). \tag{A.35}$$

Further, $H[(\tilde{\delta}_0 - \delta_0), \cdots, (\tilde{\delta}_p - \delta_p)]'$ ($H = diag[T^{1/2}, \cdots, T^{1/2+p}]$) has the same distribution as $[\tilde{\psi}_0, \cdots, \tilde{\psi}_p]'$ in the limit. Hence we obtain the desired result.

(ii) Noting that $T^{-2}\sum_{t=2}^{T} S_t S_t' \Rightarrow \int_0^1 B(r)B(r)'dr$, we obtain the result in the same way as in part (i).

(iii) Writing

$$\begin{aligned}\sum_{t=2}^{T} \bar{S}_t\bar{S}_t' &= \sum_{t=2}^{T}\{S_t - (\bar{\delta}_0 - \delta_0)\sum_{j=1}^{t-1} j^m - \cdots - (\bar{\delta}_p - \delta_p)\sum_{j=1}^{t-1} j^p\} \\ &\quad \cdot\{S_t - (\bar{\delta}_0 - \delta_0)\sum_{j=1}^{t-1} j^m - \cdots - (\bar{\delta}_p - \delta_p)\sum_{j=1}^{t-1} j^p\}' \end{aligned} \tag{A.36}$$

and noting that $H[(\bar{\delta}_0 - \delta_0), \cdots, (\bar{\delta}_p - \delta_p)]'$ has the same distribution as $[\bar{\eta}_0, \cdots, \bar{\eta}_p]'$, we obtain the desired result by using $T^{-2} \sum_{t=2}^{T} S_t S_t' \Rightarrow \int_0^1 B(r)B(r)'dr$.

(iv), (v), (vi) These are trivially obtained by applying the same methods as for Lemma A in Choi and Yu (1992) to each element of the matrices.

Proof of Theorem 2.

(a) Because $\Omega_l^{-1/2}\bar{\eta}_i = \bar{\alpha}_i$ and $\Omega_l^{-1/2}\tilde{\psi}_i = \tilde{\gamma}_i (i = 1, \cdots, n)$, we have $\Omega_l^{-1/2}\bar{B}(r) = \bar{W}(r)$ and $\Omega_l^{-1/2}\tilde{B}(r) = \tilde{W}(r)$ for any $\omega \in X$. Therefore, the required results follow from Lemma B.

(b) First, we consider the case where x_t is not cointegrated. Because $\tilde{S}_t$, $\bar{S}_t = I(m + k + 1)$ under the alternative, we obtain the results (using $\tilde{S}_t$ and $\bar{S}_t$) analogous to (A.9), (A.13) and (A.15). Further, because $\tilde{W}(r)$ and $\bar{W}(r)$ are continuous and non-differentiable *a.s.*, we may obtain the results similar to (A.10) and (A.14) by employing the same methods as for Chan and Wei (1988)' lemma 3.1.1. Next, we trivially obtain the same results for the case of cointegrated x_t by redefining the transformation matrix G as in the proof of Theorem 1 (b). Hence, the required results follow.

Proof of Theorem 3.

(i), (ii), (iii) The regression residual $\tilde{S}_t$ can be written as

$$\begin{aligned} \tilde{S}_t &= \delta_{q+1}\{\sum_{j=1}^{t} j^{q+1} - tb_{q+1} - \cdots - t^{q+1}b_1\} \\ &\quad + \cdots + \delta_p\{\sum_{j=1}^{t} j^p - tb_p - \cdots - t^{q+1}b_{p-q}\} + S_t^*, \end{aligned} \tag{A.37}$$

where $b_i = O(T^i)$ and $\{S_t^*\}$ denotes the projection of $\{S_t\}$ onto the orthogonal complement of the space spanned by $\{\sum_{j=1}^{t} j^0, \sum_{j=1}^{t} j^1, \cdots, \sum_{j=1}^{t} j^q\}$. Noting that $S_t^* = I(m+1)$, we obtain for any $i, j = 1, \cdots, n$.

$$[\sum_{t=1}^{T} \tilde{S}_t \tilde{S}_t']^{(i,j)} = O_p(T^{2p+3}), \tag{A.38}$$

$$[\sum_{t=1}^{T} \Delta \tilde{S}_t \tilde{S}_{t-1}']^{(i,j)} = O_p(T^{2p+2}), \tag{A.39}$$

$$[\tilde{\Omega}_l]^{(i,j)} = O_p(T^{\delta+2p}), \tag{A.40}$$

$$[\tilde{\Omega}_1]^{(i,j)} = O_p(T^{2p}), \tag{A.41}$$

from which the required results follow. Note the result regarding $\tilde{\Omega}_l$ is obtained from equation (3.3) in Section 3.

(iv) The regression residual $\bar{x}_t$ can be written as

$$\begin{aligned} \bar{x}_t &= \delta_{q+1}\{t^{q+1} - t^0 a_{q+1} - \cdots - t^q a_1\} + \\ & \cdots + \delta_p\{t^p - t^0 a_p - \cdots - t^q a_{p-q}\} + x_t^*, \end{aligned} \tag{A.42}$$

where $a_i = O(T^i)$ and $\{x_t^*\}$ denotes the projection of $\{x_t\}$ onto the orthogonal complement of the space spanned by $\{t^0, t^1, \cdots, t^q\}$. Also note that $x_t^* = I(m)$. Using (A.42), we have

$$\begin{aligned} \bar{S}_t &= \delta_{q+1}\{\sum_{j=1}^{t} j^{q+1} - \sum_{j=1}^{t} j^0 a_{q+1} - \cdots - \sum_{j=1}^{t} j^q a_1\} \\ & + \cdots + \delta_p\{\sum_{j=1}^{t} j^p - \sum_{j=1}^{j} j^0 a_p - \cdots - \sum_{j=1}^{t} j^q a_{p-q}\} + \sum_{j=1}^{t} x_j^*. \end{aligned} \tag{A.43}$$

Therefore, we obtain for any $i, j = 1, \cdots, n$,

$$[\sum_{t=1}^{T} \bar{S}_t \bar{S}_t']^{(i,j)} = O_p(T^{2p+3}), \tag{A.44}$$

$$[\sum_{t=1}^{T} \Delta \bar{S}_t \bar{S}'_{t-1}]^{(i,j)} = O_p(T^{2p+2})(m \geq 1), \tag{A.45}$$

$$[\bar{\Omega}_l]^{(i,j)} = O_p(T^{\delta+2p}), \tag{A.46}$$

from which the results follow.

Appendix B

Proofs for Chapter III

For the proofs in this section, we will use the weak convergence results in Phillips and Durlauf (1986), Park and Phillips (1988), Phillips (1987) and Kwiatkowski, Phillips, Schmidt and Shin (1992) freely without referring to these articles in each instance.

Proof of Lemma 1.

This is deduced from Park's (1992) Theorem 4.1.

Proof of Lemma 2.

We make the simplifying assumptions $x_0 = 0$ and $c_0 = 0$ without loss of generality for the asymptotic distribution of $\bar{B}$. Letting $F_T = diag[T^{1/2}, T^{1+1/2}, \cdots, T^{p+1/2}]$ and denoting the consistent estimates of Ω_{12} and Ω_{22} using $\{\tilde{u}_t\}$ as $\tilde{\Omega}_{12}$ and $\tilde{\Omega}_{22}$, respectively, we obtain

$$\begin{aligned}
\sum_{t=1}^{T} \bar{u}_t c_t' F_T^{-1} &= \sum_{t=1}^{T} (u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}\Delta x_t) c_t' F_T^{-1} - (\tilde{A} - A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)' \sum_{t=1}^{T} \tilde{w}_t c_t' F_T^{-1} \\
&= \sum_{t=1}^{T} (u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}\Delta x_t) c_t' F_T^{-1} + O_p(T^{-1}) \\
&\Rightarrow \int_0^1 dB_{1.2}(r) R(r)', \ R(r) = [1, r, \cdots, r^p]', \qquad \text{(B.1)}
\end{aligned}$$

because $\tilde{A} - A = O_p(T^{-1})$ and $\sum_{t=1}^{T} \tilde{w}_t c_t' F_T^{-1} = O_p(1)$. Further,

$$T^{-1} \sum_{t=1}^{T} \bar{u}_t \bar{x}_t' = T^{-1} \sum_{t=1}^{T} \{u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}\Delta x_t - (\tilde{A} - A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{w}_t\}\{x_t - (\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{w}_t\}'$$

$$
\begin{aligned}
&= T^{-1}\sum_{t=1}^{T}(u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}\Delta x_t)x_t' - T^{-1}\sum_{t=1}^{T}\tilde{\kappa}\begin{bmatrix} u_t \\ \Delta x_t \end{bmatrix}\tilde{w}_t'\tilde{\Lambda}^{-1}\tilde{\Gamma}_2' && \text{(B.2)} \\
&\quad +O_p(T^{-1}) \\
&= A_T - B_T + O_p(T^{-1}),\ say, && \text{(B.3)}
\end{aligned}
$$

where $\tilde{\kappa} = [I - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}]$. But

$$
\begin{aligned}
A_T &= T^{-1}\sum_{t=1}^{T}(u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}\Delta x_t)x_t' \\
&\Rightarrow \int_0^1 dB_{1.2}(r)B_2(r)'dr + \Gamma_{12}' - \Omega_{12}\Omega_{22}^{-1}\Gamma_{22}. && \text{(B.4)}
\end{aligned}
$$

In addition,

$$
B_T \xrightarrow{p} [I - \Omega_{12}\Omega_{22}^{-1}]\Gamma_2' = \Gamma_{12}' - \Omega_{12}\Omega_{22}^{-1}\Gamma_{22}. \tag{B.5}
$$

Therefore, (B.4) and (B.5) yield

$$
T^{-1}\sum_{t=1}^{T}\bar{u}_t\bar{x}_t' \Rightarrow \int_0^1 dB_{1.2}(r)B_2(r)'. \tag{B.6}
$$

Writing $\sum_{t=1}^{T}\bar{u}_t\bar{q}_t' = [(\sum_{t=1}^{T}\bar{u}_t c_t')'(\sum_{t=1}^{T}\bar{u}_t\bar{x}_t')']'$ and using (B.1) and (B.6), we obtain

$$
\sum_{t=1}^{T}\bar{u}_t\bar{q}_t'D_T \Rightarrow \int_0^1 dB_{1.2}(r)Q(r)'. \tag{B.7}
$$

Because $\bar{x}_t$ behaves as if it were x_t in the limit, it is straightforward to establish

$$
D_T^{-1}\sum_{t=1}^{T}\bar{q}_t\bar{q}_t'D_T^{-1} \Rightarrow \int_0^1 Q(r)Q(r)'dr. \tag{B.8}
$$

Now the required result follows from (B.7) and (B.8).

Proof of Lemma 3.

We assume $x_0 = 0$ and $c_0 = 0$ as in the proof of Lemma 2. Letting $G_T = diag[T^{2+1/2}, T^{3+1/2}, \cdots, T^{p+2+1/2}]$, $S^c_t = \sum_{i=1}^t c_i$, $S^u_t = \sum_{i=1}^t u_i$, and $\tilde{S}^w_t = \sum_{i=1}^t \tilde{w}_i$, we have

$$\begin{aligned}\sum_{t=1}^{T} \bar{S}^u_t S^{c\prime}_t G_T^{-1} &= \sum_{t=1}^{T} \{S^u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1} x_t - (\tilde{A} - A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{S}^w_t\} S^{c\prime}_t G_T^{-1} \\ &= \sum_{t=1}^{T} (S^u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1} x_t) S^{c\prime}_t G_T^{-1} + O_p(T^{-1}) \\ &\Rightarrow \int_0^1 B_{1.2}(r)\{\int_0^r R(s)'ds\}dr, \; R(s) = [1, s, \cdots, s^p]'. \qquad \text{(B.9)}\end{aligned}$$

Note that $\sum_{t=1}^T \tilde{S}^w_t S^{c\prime}_t G_T^{-1} = O_p(1)$. Denoting $\bar{S}^x_t = \sum_{i=1}^t \bar{x}_i$ and $S^x_t = \sum_{i=1}^t x_i$, we may write

$$\begin{aligned}\frac{1}{T^3}\sum_{t=1}^{T} \bar{S}^u_t \bar{S}^{x\prime}_t &= \frac{1}{T^3}\sum_{t=1}^{T} \{S^u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1} x_t - (\tilde{A} - A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{S}^w_t\}\{S^x_t - (\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{S}^w_t\}' \\ &= \frac{1}{T^3}\sum_{t=1}^{T} (S^u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1} x_t) S^{x\prime}_t - \frac{1}{T^3}\sum_{t=1}^{T} \tilde{\kappa} \begin{bmatrix} S^u_t \\ x_t \end{bmatrix} \tilde{S}^{w\prime}_t \tilde{\Lambda}^{-1}\tilde{\Gamma}_2' \\ &\quad + O_p(T^{-1}), \\ &= K_T - L_T + O_p(T^{-1}), \text{ say,} \qquad \text{(B.10)}\end{aligned}$$

where $\tilde{\kappa} = [I - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}]$. Note that $\sum_{t=1}^T \tilde{S}^w_t S^x_t = O_p(T^3)$ and $\sum_{t=1}^T \tilde{S}^w_t \tilde{S}^w_t = O_p(T^2)$. But

$$\begin{aligned}K_T &= T^{-3}\sum_{t=1}^{T} (S^u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1} x_t) S^{x\prime}_t \\ &\Rightarrow \int_0^1 B_{1.2}(r)(\int_0^r B_2(s)'ds)dr. \qquad \text{(B.11)}\end{aligned}$$

Further,

$$L_T = O_p(T^{-1}), \qquad \text{(B.12)}$$

because $\sum_{t=1}^{T} \begin{bmatrix} S_t^u \\ x_t \end{bmatrix} S_t^{w\prime} = O_p(T^2)$. Therefore, it follows from (B.11) and (B.12) that

$$T^{-3} \sum_{t=1}^{T} \bar{S}_t^u \bar{S}_t^{x\prime} \Rightarrow \int_0^1 B_{1.2}(r) (\int_0^r B_2(s)' ds) dr. \tag{B.13}$$

Writing $\sum_{t=1}^{T} \bar{S}_t^u \bar{S}_t^{q\prime} = [(\sum_{t=1}^{T} \bar{S}_t^u S_t^{c\prime})'(\sum_{t=1}^{T} \bar{S}_t^u \bar{S}_t^{x\prime})']'$ and using (B.9) and (B.13), we obtain

$$\sum_{t=1}^{T} \bar{S}_t^u \bar{S}_t^{q\prime} H_T^{-1} \Rightarrow \int_0^1 B_{1.2}(r) S(r)' dr, \tag{B.14}$$

where $H_T = diag[T^{2+1/2}, T^{3+1/2}, \cdots, T^{p+2+1/2}, T^3, \cdots, T^3]$. In addition, it is straightforward to establish

$$H_T^{-1} \sum_{t=1}^{T} \bar{S}_t^q \bar{S}_t^{q\prime} D_T^{-1} \Rightarrow \int_0^1 S(r) S(r)' dr. \tag{B.15}$$

Therefore, the required result follows from (B.14) and (B.15).

Proof of Theorem 1.

(a) Write $y_t^* - A^* x_t^* = u_t - \hat{\Omega}_{12} \hat{\Omega}_{22}^{-1} \Delta x_t - (\hat{A} - A)(\hat{\Lambda}^{-1} \hat{\Gamma}_2)' \hat{w}_t - (A^* - A) x_t^*$. We assume $x_0 = 0$ without loss of generality for the asymptotic distributions we are to derive. Denoting $S_t^{x*} = \sum_{i=1}^{t} x_i^*$, $\hat{S}_t^w = \sum_{i=1}^{t} \hat{w}_i$ and $\bar{B}_2(r) = \int_0^r B_2(s) ds$, we obtain by using Lemma 1

$$\begin{aligned}
T^{-1} \sum_{t=2}^{T} S_{t-1}^* \Delta S_t^{*\prime} &= T^{-1} \sum_{t=2}^{T} \{S_{t-1}^u - \hat{\Omega}_{12} \hat{\Omega}_{22}^{-1} x_{t-1} - (\hat{A} - A)(\hat{\Lambda}^{-1} \hat{\Gamma}_2)' \hat{S}_{t-1}^w \\
&\quad -(A^* - A) S_{t-1}^{x*}\} \{u_t - \hat{\Omega}_{12} \hat{\Omega}_{22}^{-1} \Delta x_t - (A^* - A) x_t^*\}' \\
&= T^{-1} \sum_{t=2}^{T} \{S_{t-1}^u - \hat{\Omega}_{12} \hat{\Omega}_{22}^{-1} x_{t-1} - (A^* - A) S_{t-1}^x\} \\
&\quad \cdot \{u_t - \hat{\Omega}_{12} \hat{\Omega}_{22}^{-1} \Delta x_t - (A^* - A) x_t\}' + O_p(T^{-1}) \\
&\Rightarrow \int_0^1 [B_{1.2}(r) - \int_0^1 dB_{1.2}(s) B_2(s)' \{\int_0^1 B_2(s) B_2(s)' ds\}^{-1} \bar{B}_2(r)]
\end{aligned}$$

$$
\begin{aligned}
&\cdot[dB_{1.2}(r) - \int_0^1 dB_{1.2}(s)B_2(s)'\{\int_0^1 B_2(s)B_2(s)'ds\}^{-1}B_2(r)dr]' \\
&+\kappa\Sigma\kappa' \\
\equiv\ & \Omega_{11.2}^{1/2}\int_0^1[W_1(r) - \int_0^1 dW_1(s)W_2(s)'\{\int_0^1 W_2(s)W_2(s)'ds\}^{-1} \\
&\bar{W}_2(r)][dW_1(r) - \int_0^1 dW_1(s)W_2(s)'\{\int_0^1 W_2(s)W_2(s)'ds\}^{-1} \\
&W_2(r)dr]'\Omega_{11.2}^{1/2} + \kappa\Sigma\kappa' \qquad \text{(B.16)}
\end{aligned}
$$

and

$$
\begin{aligned}
T^{-2}\sum_{t=1}^{T} S_t^* S_t^{*\prime} &= T^{-2}\sum_{t=1}^{T}\{S_t^u - \hat{\Omega}_{12}\hat{\Omega}_{22}^{-1}x_t - (\hat{A} - A)(\hat{\Lambda}^{-1}\hat{\Gamma}_2)'\hat{S}_t^w - (A^* - A)S_t^{x*}\} \\
&\quad\cdot\{S_t^u - \hat{\Omega}_{12}\hat{\Omega}_{22}^{-1}x_t - (\hat{A} - A)(\hat{\Lambda}^{-1}\hat{\Gamma}_2)'\hat{S}_t^w - (A^* - A)S_t^{x*}\}' \\
&= T^{-2}\sum_{t=1}^{T}\{S_t - \hat{\Omega}_{12}\hat{\Omega}_{22}^{-1}x_t - (A^* - A)S_t^x\} \\
&\quad\cdot\{S_t - \hat{\Omega}_{12}\hat{\Omega}_{22}^{-1}x_t - (A^* - A)S_t^x\}' + O_p(T^{-1}) \\
&\Rightarrow \int_0^1[B_{1.2}(r) - \int_0^1 dB_{1.2}(s)B_2(s)'\{\int_0^1 B_2(s)B_2(s)'ds\}^{-1}\bar{B}_2(r)] \\
&\quad\cdot[B_{1.2}(r) - \int_0^1 dB_{1.2}(s)B_2(s)'\{\int_0^1 B_2(s)B_2(s)'ds\}^{-1}\bar{B}_2(r)]'dr \\
&\equiv \Omega_{11.2}^{1/2}\int_0^1[W_1(r) - \int_0^1 dW_1(s)W_2(s)'\{\int_0^1 W_2(s)W_2(s)'ds\}^{-1} \\
&\quad\bar{W}_2(r)][W_1(r) - \int_0^1 dW_1(s)W_2(s)'\{\int_0^1 W_2(s)W_2(s)'ds\}^{-1} \\
&\quad\bar{W}_2(r)]'dr\Omega_{11.2}^{1/2}. \qquad \text{(B.17)}
\end{aligned}
$$

Note that we obtain the equivalence (in distribution) relations from $\Omega_{11.2}^{-1/2}B_{1.2}(r) \equiv W_1(r)$ and $\Omega_{22}^{-1/2}B_2(r) \equiv W_2(r)$. Because $\Omega_{11.2}^{*} \xrightarrow{p} \Omega_{11.2}$, the required results follow from (B.16) and (B.17).

(*b*) Write $\bar{y}_t - \bar{B}\bar{q}_t = u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}\Delta x_t - (\tilde{A} - A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{w}_t - (\bar{B} - B)\bar{q}_t$, and $\bar{S}_t^y - \check{B}\bar{S}_t^q =$

$S_t^u - \tilde{\Omega}_{12}\tilde{\Omega}_{22}(x_t - x_0) - (\tilde{A} - A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{S}_t^w - (\check{B} - B)\bar{S}_t^q$. Without loss of generality, we assume $x_0 = 0$. We obtain by using Lemma 2

$$\begin{aligned}
T^{-1}\sum_{t=2}^{T}\check{S}_{t-1}\Delta\check{S}_t' \quad &= \quad T^{-1}\sum_{t=2}^{T}\{S_{t-1}^u - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}x_{t-1} - (\tilde{A} - A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{S}_{t-1}^w \\
&\quad -(\check{B} - B)\bar{S}_{t-1}^q\}\{u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}\Delta x_t \\
&\quad -(\tilde{A} - A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2'\tilde{w}_t - (\check{B} - B)\bar{q}_t\}' \\
&= \quad T^{-1}\sum_{t=2}^{T}\{S_{t-1}^u - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}x_{t-1} - (\check{B} - B)S_{t-1}^q\} \\
&\quad \cdot\{u_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}\Delta x_t - (\check{B} - B)q_t\}' + O_p(T^{-1}) \\
&\Rightarrow \quad \int_0^1 [B_{1.2}(r) - \int_0^1 B_{1.2}(s)S(s)'ds\{\int_0^1 S(s)S(s)'ds\}^{-1}S(r)] \\
&\quad \cdot[dB_{1.2}(r) - \int_0^1 B_{1.2}(s)S(s)'ds\{\int_0^1 S(s)S(s)'ds\}^{-1}Q(r)'dr \\
&\quad +\kappa\Sigma\kappa' \\
&\equiv \quad \Omega_{11.2}^{1/2}\int_0^1 [W_1(r) - \int_0^1 W_1(s)S_w(s)'ds\{\int_0^1 S_w(s)S_w(s)'ds\}^{-1} \\
&\quad S_w(r)][dW_1(r) - \int_0^1 W_1(s)S_w(s)'ds\{\int_0^1 S_w(s)S_w(s)'ds\}^{-1} \\
&\quad Q_w(r)'dr]\Omega_{11.2}^{1/2} + \kappa\Sigma\kappa', \qquad \text{(B.18)}
\end{aligned}$$

because $ZS(s) \equiv S_w(s)$ where $Z = \begin{bmatrix} I & 0 \\ 0 & \Omega_{22}^{-1} \end{bmatrix}$. In the same way, we obtain

$$\begin{aligned}
T^{-2}\sum_{t=1}^{T}\check{S}_t\check{S}_t' \quad &= \quad T^{-2}\sum_{t=1}^{T}\{S_t^u - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}x_t - (\tilde{A} - A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{S}_t^w - (\check{B} - B)\bar{S}_t^q\} \\
&\quad \cdot\{S_t^u - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}x_t - (\tilde{A} - A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{S}_t^w - (\check{B} - B)\bar{S}_t^q\}' \\
&= \quad T^{-2}\sum_{t=1}^{T}\{S_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}x_t - (\check{B} - B)S_t^q\} \\
&\quad \cdot\{S_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}x_t - (\check{B} - B)S_t^q\}' + O_p(T^{-1}) \\
&\Rightarrow \quad \int_0^1 [B_{1.2}(r) - \int_0^1 B_{1.2}(s)S(s)'ds\{\int_0^1 S(s)S(s)'ds\}^{-1}S(r)]
\end{aligned}$$

$$
\begin{aligned}
& \cdot[B_{1.2}(r) - \int_0^1 B_{1.2}(s)S(s)'ds\{\int_0^1 S(s)S(s)'ds\}^{-1}S(r)']dr \\
\equiv\; & \Omega_{11.2}^{1/2}\int_0^1 [W_1(r) - \int_0^1 W_1(s)S_w(s)'ds\{\int_0^1 S_w(s)S_w(s)'ds\}^{-1} \\
& S_w(r)][W_1(r) - \int_0^1 W_1(s)S_w(s)'ds\{\int_0^1 S_w(s)S_w(s)'ds\}^{-1} \\
& S_w(r)'dr]\Omega_{11.2}^{1/2} \qquad \text{(B.19)}
\end{aligned}
$$

and

$$
\begin{aligned}
T^{-2}\sum_{t=1}^{T}\bar{S}_t\bar{S}_t' \;=\; & T^{-2}\sum_{t=1}^{T}\{S_t^u - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}x_t - (\tilde{A}-A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{S}_t^w - (\bar{B}-B)\bar{S}_t^q\} \\
& \cdot\{S_t^u - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}x_t - (\tilde{A}-A)(\tilde{\Lambda}^{-1}\tilde{\Gamma}_2)'\tilde{S}_t^w - (\bar{B}-B)\bar{S}_t^q\}' \\
=\; & T^{-2}\sum_{t=1}^{T}\{S_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}x_t - (\bar{B}-B)S_t^q\} \\
& \cdot\{S_t - \tilde{\Omega}_{12}\tilde{\Omega}_{22}^{-1}x_t - (\bar{B}-B)S_t^q\}' + O_p(T^{-1}) \\
\Rightarrow\; & \int_0^1 [B_{1.2}(r) - \int_0^1 dB_{1.2}(s)Q(s)'\{\int_0^1 Q(s)Q(s)'ds\}^{-1}S(r)] \\
& \cdot[B_{1.2}(r) - \int_0^1 dB_{1.2}(s)Q(s)'(\int_0^1 Q(s)Q(s)'ds)^{-1}S(r)]'dr \\
\equiv\; & \Omega_{11.2}^{1/2}\int_0^1 [W_1(r) - \int_0^1 dW_1(s)Q_w(s)'ds\{\int_0^1 Q_w(s)Q_w(s)'ds\}^{-1} \\
& S_w(r)][W_1(r) - \int_0^1 dW_1(s)Q_w(s)'ds\{\int_0^1 Q_w(s)Q_w(s)'ds\}^{-1} \\
& S(r)]'dr\Omega_{11.2}^{1/2}. \qquad \text{(B.20)}
\end{aligned}
$$

Because $\check{\Omega}_{11.2} \xrightarrow{p} \Omega_{11.2}$ and $\bar{\Omega}_{11.2} \xrightarrow{p} \Omega_{11.2}$, we obtain the required results from (B.18), (B.19) and (B.20).

Proof of Theorem 2.

(a) Consider the system of equation (3.1). Under the alternative, there is at least one equation that is spurious in nature. Without loss of generality, arrange the first n_1

equations to be spurious and the remaining $n_2 = n - n_1$ equations to be cointegrated. Then, we have for the OLS estimate of A

$$N_T(\hat{A} - A) = O_p(1), \tag{B.21}$$

$N_T = diag[1, \cdots, 1, T, \cdots, T]$. Further, write

$$\begin{aligned} \sum_{t=1}^{T} x_t^* x_t^{*\prime} &= \sum_{t=1}^{T} x_t x_t' - (\hat{\Lambda}^{-1}\hat{\Gamma}_2)' \sum_{t=1}^{T} \hat{w}_t x_t' - \sum_{t=1}^{T} x_t \hat{w}_t' (\hat{\Lambda}^{-1}\hat{\Gamma}_2) \\ &\quad + (\hat{\Lambda}^{-1}\hat{\Gamma}_2)' \sum_{t=1}^{T} \hat{w}_t \hat{w}_t' (\hat{\Lambda}^{-1}\hat{\Gamma}_2). \end{aligned} \tag{B.22}$$

But $M_T \hat{\Lambda} M_T^{-1} = O_p(1)$, where $M_T = diag[T^{1/2}, \cdots, T^{1/2}, 1, \cdots, 1]$,

$$\hat{\Gamma}_2 = \begin{array}{cc} m & \\ \left[\begin{array}{c} O_p(T^\delta) \\ O_p(1) \end{array}\right] & \begin{array}{c} n_1 \\ n_2 + m \end{array} \end{array}, \tag{B.23}$$

$$\sum_{t=1}^{T} \hat{w}_t x_t' = \begin{array}{cc} n + m & \\ \left[\begin{array}{c} O_p(T^2) \\ O_p(T) \end{array}\right] & \begin{array}{c} n_1 \\ n_2 + m \end{array} \end{array}, \tag{B.24}$$

$$T^{-1} M_T^{-1} \sum_{t=1}^{T} \hat{w}_t \hat{w}_t' M_T^{-1} = O_p(1). \tag{B.25}$$

Note that we may show that the matrix $M_T \hat{\Lambda} M_T^{-1}$ is nonsingular in the limit *a.s.* by using the same methods as for Chan and Wei's (1988) Lemma 3.1.1 and Lemma A in Appendix A. Therefore, it follows that

$$T^{-2} \sum_{t=1}^{T} x_t^* x_t^{*\prime} = T^{-2} \sum_{t=1}^{T} x_t x_t' + o_p(1). \tag{B.26}$$

In addition, writing

$$\begin{aligned} \sum_{t=1}^{T} u_t^* x_t^{*\prime} &= \sum_{t=1}^{T} u_t x_t - \sum_{t=1}^{T} u_t \hat{w}_t' \hat{\Lambda}^{-1} \hat{\Gamma}_2 - \hat{\Omega}_{12} \hat{\Omega}_{22}^{-1} (\sum_{t=1}^{T} \Delta x_t x_t' - \sum_{t=1}^{T} \Delta x_t \hat{w}_t' \hat{\Lambda}^{-1} \hat{\Gamma}_2) \\ &\quad - (\hat{A} - A)(\hat{\Lambda}^{-1}\hat{\Gamma}_2)' (\sum_{t=1}^{T} \hat{w}_t x_t' - \sum_{t=1}^{T} \hat{w}_t \hat{w}_t' \hat{\Lambda}^{-1} \hat{\Gamma}_2) \end{aligned} \tag{B.27}$$

and using

$$\sum_{t=1}^{T} u_t \hat{w}_t' = \begin{array}{cc} \begin{array}{cc} n_1 & n_2+m \end{array} & \\ \left[\begin{array}{cc} O_p(T^2) & O_p(T) \\ O_p(T) & O_p(T) \end{array}\right] & \begin{array}{c} n_1 \\ n_2 \end{array} \end{array}, \tag{B.28}$$

$$\hat{\Omega}_{12} = \begin{array}{cc} m & \\ \left[\begin{array}{c} O_p(T^{\delta}) \\ O_p(1) \end{array}\right] & \begin{array}{c} n_1 \\ n_2 \end{array} \end{array}, \tag{B.29}$$

$$\hat{\Omega}_{22} = O_p(1), \ \sum_{t=1}^{T} \Delta x_t x_t' = O_p(T), \ \sum_{t=1}^{T} \Delta x_t \hat{w}_t' = O_p(T), \tag{B.30}$$

and

$$\hat{A} - A = \begin{array}{cc} m & \\ \left[\begin{array}{c} O_p(1) \\ O_p(T^{-1}) \end{array}\right] & \begin{array}{c} n_1 \\ n_2 \end{array} \end{array}, \tag{B.31}$$

we obtain

$$\sum_{t=1}^{T} u_t^* x_t^{*\prime} = \begin{array}{cc} m & \\ \left[\begin{array}{c} O_p(T^2) \\ O_p(T) \end{array}\right] & \begin{array}{c} n_1 \\ n_2 \end{array} \end{array}. \tag{B.32}$$

From (B.26) and (B.32), it follows that

$$N_T(\hat{A}^* - A) = O_p(1). \tag{B.33}$$

Next, write

$$S_t^* = S_t^u - \hat{\Omega}_{12}\hat{\Omega}_{22}^{-1}x_t - (\hat{A} - A)(\hat{\Lambda}^{-1}\hat{\Gamma}_2)'\hat{S}_t^w - (A^* - A)(S_t^x - \hat{\Gamma}_2'\hat{\Lambda}^{-1}\hat{S}_t^w) \tag{B.34}$$

and let $P_T = diag[T, \cdots, T, 1, \cdots, 1]$. Then, using (B.21), (B.33) and the same methods as for (B.26) and (B.32) gives

$$\begin{aligned} T^{-2}P_T^{-1}\sum_{t=1}^{T} S_t^* S_t^{*\prime} P_T^{-1} &= T^{-2}P_T^{-1}\sum_{t=1}^{T} S_t^u S_t^{u\prime} P_T^{-1} + T^{-2}P_T^{-1}\sum_{t=1}^{T} S_t^u S_t^{x\prime}(A^* - A)'P_T^{-1} \\ &\quad + T^{-2}P_T^{-1}(A^* - A)\sum_{t=1}^{T} S_t^x S_t^{x\prime}(A^* - A)'P_T^{-1} + o_p(1) \\ &= O_p(1). \end{aligned} \tag{B.35}$$

In the same way,

$$T^{-1}P_T^{-1}\sum_{t=2}^{T}\Delta S_t^* S_{t-1}^{*'} P_T^{-1} = O_p(1). \tag{B.36}$$

Further,

$$Q_T^{-1}P_T^{-1}\Omega_{11.2}^* P_T^{-1}Q_T^{-1} = O_p(1), \tag{B.37}$$

and

$$Q_T^{-1}P_T^{-1}\hat{\kappa}\Sigma\hat{\kappa}' P_T^{-1}Q_T^{-1} = O_p(1), \tag{B.38}$$

where $Q_T = diag[T^{(\delta-1)/2}, \cdots, T^{(\delta-1)/2}, 1, \cdots, 1]$. We may show the *a.s.* nonsingularity of T^{-2} P_T^{-1} $\sum_{t=1}^{T}$ $S_t^* S_t^{*'} P_T^{-1}$ in the limit by using the same methods as for Chan and Wei's (1988) Lemma 3.1.1. Additionally, the *a.s.* nonsingularity of Q_T^{-1} P_T^{-1} $\Omega_{11.2}^*$ P_T^{-1} Q_T^{-1} in the limit is obtained by using the same methods as for the *a.s.* nonsingularity of M_T $\hat{\Lambda}$ M_T^{-1}. Hence, rewriting the test statistics as

$$\begin{aligned} LM_I \;=\;& tr\{Q_T^{-1}(T^{-1}P_T^{-1}\sum_{t=2}^{T}\Delta S_t^* S_{t-1}^{*'} P_T^{-1} - P_T^{-1}\hat{\kappa}\Sigma'\hat{\kappa}' P_T^{-1})Q_T^{-1} \\ & \cdot(Q_T^{-1}P_T^{-1}\Omega_{11.2}^* P_T^{-1}Q_T^{-1})^{-1} \\ & \cdot Q_T^{-1}(T^{-1}P_T^{-1}\sum_{t=2}^{T} S_{t-1}^*\Delta S_t^{*'} P_T^{-1} - P_T^{-1}\hat{\kappa}\Sigma\hat{\kappa}' P_T^{-1})Q_T^{-1} \\ & \cdot(Q_T^{-1}P_T^{-1}\Omega_{11.2}^* P_T^{-1}Q_T^{-1})^{-1}\}, \end{aligned} \tag{B.39}$$

$$\begin{aligned} LM_{II} \;=\;& tr\{Q_T^{-1}(T^{-1}P_T^{-1}\sum_{t=2}^{T}\Delta S_t^* S_{t-1}^{*'} P_T^{-1} - P_T^{-1}\hat{\kappa}\Sigma'\hat{\kappa}' P_T^{-1}) \\ & \cdot(T^{-2}P_T^{-1}\sum_{t=2}^{T} S_{t-1}^* S_{t-1}^{*'} P_T^{-1})^{-1} \\ & \cdot(T^{-1}P_T^{-1}\sum_{t=2}^{T} S_{t-1}^*\Delta S_t^{*'} P_T^{-1} - P_T^{-1}\hat{\kappa}\Sigma\hat{\kappa}' P_T^{-1}) \\ & \cdot Q_T^{-1}(Q_T^{-1}P_T^{-1}\Omega_{11.2}^* P_T^{-1}Q_T^{-1})^{-1}\}, \end{aligned} \tag{B.40}$$

$$SBDH = tr\{Q_T^{-1}(T^{-2}\sum_{t=1}^{T} P_T^{-1}S_t^* S_t^{*\prime} P_T^{-1})Q_T^{-1}(Q_T^{-1}P_T^{-1}\Omega_{11.2}^* P_T^{-1}Q_T^{-1})^{-1}\} \qquad (B.41)$$

and using (B.35)-(B.38), we obtain the desired results.

(*b*) Because the moment based on a raw series and that on the detrended series share the same probabilistic order of magnitude, the test statistics using $\bar{S}_t$ and $\check{S}_t$ diverge at the same rates as those using S_t^*. Hence, the required results follow.

Appendix C

Proofs for Chapter IV

Proof of Theorem 1.

The true DGP is given by

$$\begin{aligned} y_t &= A_1 c_t \iota_1 + A_2 c_t \iota_2 + x_t, \\ &= A_1 c_t + (A_2 - A_1) c_t \iota_2 + x_t, \end{aligned} \tag{C.1}$$

whereas a researcher runs the following regression equation to estimate the residuals:

$$y_t = A_1 c_t + x_t. \tag{C.2}$$

The residual from (C.2) is given by

$$\begin{aligned} \bar{x}_t &= x_t - \sum x_t c_t' (\sum c_t c_t')^{-1} c_t + (A_2 - A_1) c_t \iota_2 \\ &\quad -(A_2 - A_1) \sum \iota_2 c_t c_t' (\sum c_t c_t')^{-1} c_t \end{aligned} \tag{C.3}$$

and

$$\begin{aligned} \bar{S}_t &= S_t - \sum x_t c_t' (\sum c_t c_t')^{-1} S_t^c + (A_2 - A_1) S_t^{c\iota_2} \\ &\quad -(A_2 - A_1) \sum \iota_2 c_t c_t' (\sum c_t c_t')^{-1} S_t^c \\ &= E_{1t} - E_{2t} + E_{3t} - E_{4t}, \end{aligned} \tag{C.4}$$

where $S_t^w = \sum_{i=1}^t w_i$. Note that $\delta_T^{-1}\, c_{[Tr]} \rightarrow c(r)$, $T^{-1}\, \delta_T^{-1}\, S_{[Tr]}^c \rightarrow \int_0^r c(s)ds$, and T^{-1} $\delta_T^{-1}\, S_{[Tr]}^{c\iota_2} \rightarrow \int_0^r c\iota_2(s)ds$. Also, $T^{-1/2}\, E_{1[Tr]} \Rightarrow B(r)$, $T^{-1/2}\, E_{2[Tr]} \Rightarrow \int_0^1 dBc'\, (\int_0^1 cc')^{-1}$ $\bar{c}(r)$, and $T^{-1/2}\, E_{3[Tr]} = O_p(T^{p+1/2}) = T^{-1/2}E_{4[Tr]}$. Since $\bar{S}_t$ is dominated by E_3 and E_4 of $O_p(T^{p+1/2})$, letting $E_t = E_{3t} + E_{4t}$, we have

$$\begin{aligned} T^{-2}\sum_{t=1}^{T} \bar{S}_t\bar{S}_t' &= T^{-2}\sum_{t=1}^{T} E_tE_t' + O_p(T^{p+1/2}) \\ &= O_p(T^{2p+1}) + O_p(T^{p+1/2}). \end{aligned} \tag{C.5}$$

Further, by KPSS (1992) and Appendix A, the longrun covariance matrix estimate is given by

$$\begin{aligned} \bar{\Omega}_l &= \sum_{h=-l}^{l} \bar{C}(h)k(h/l) \\ &= O_p(T^{2p+\delta}). \end{aligned} \tag{C.6}$$

From (C.5) and (C.6) the result (iv) follows. As for (i) to (iii) could be shown easily by applying the same method used to prove (i). For further details, see Appendix A.

Proof of Theorem 2.

Part (a): The true DGP is represented by

$$y_t = Ad_t + x_t \text{ for the bar case,} \tag{C.7}$$

and

$$S_t^y = Ah_t + S_t \text{ for the tilde case.} \tag{C.8}$$

Define weight matrix

$$\delta_T = diag[T^0, T, \cdots, T^{k-1}, T^k, T^k, \cdots, T^\ell, T^\ell, T^{\ell+1}, \cdots, T^p]. \tag{C.9}$$

Then, $\delta_T^{-1} d_{[Tr]} \rightarrow d$ and $T^{-1}\delta_T^{-1} h_t \rightarrow h = \int_0^r d(s)ds$. The limiting distributions for the OLS estimators from equations (C.7) and (C.8) are given by

$$\begin{aligned} T^{1/2}(\bar{A} - A)\delta_T &\Rightarrow \int_0^1 dBd'(\int_0^1 dd')^{-1} \\ &= \bar{F}, \end{aligned} \tag{C.10}$$

$$\begin{aligned} T^{1/2}(\tilde{A} - A)\delta_T &\Rightarrow \int_0^1 Bh'(\int_0^1 hh')^{-1} \\ &= \tilde{F}. \end{aligned} \tag{C.11}$$

Now consider the moment matrix of OLS residuals $\bar{S}_t = \sum_{i=1}^t \bar{x}_t$ and $\tilde{S}_t$.

$$\begin{aligned} T^{-2}\sum_{t=1}^T \bar{S}_t\bar{S}_t' &= T^{-2}\sum_{t=1}^T (S_t - (\bar{A} - A)h_t)(S_t - (\bar{A} - A)h_t)' \\ &\Rightarrow \int_0^1 (B - \bar{F}h)(B - \bar{F}h)', \end{aligned} \tag{C.12}$$

$$\begin{aligned} T^{-2}\sum_{t=1}^T \tilde{S}_t\tilde{S}_t' &= T^{-2}\sum_{t=1}^T (S_t - (\tilde{A} - A)h_t)(S_t - (\tilde{A} - A)h_t)' \\ &\Rightarrow \int_0^1 (B - \tilde{F}h)(B - \tilde{F}h)', \end{aligned} \tag{C.13}$$

and

$$\begin{aligned} T^{-1}\sum_{t=2}^T \Delta\tilde{S}_t\tilde{S}_{t-1}' &= T^{-1}\sum_{t=2}^T (\Delta S_t - (\tilde{A} - A)d_t)(S_{t-1} - (\tilde{A} - A)h_{t-1})' \\ &\Rightarrow \int_0^1 d(B - \tilde{F}h)(B - \tilde{F}h)' + \Omega_1'. \end{aligned} \tag{C.14}$$

Next, the covariance matrices are consistent (see Appendix A for proof)

$$\bar{\Omega}_l, \tilde{\Omega}_l, \text{ and, } \tilde{\Omega}_1 \xrightarrow{p} \Omega_l \text{ and } \Omega_1, \text{ respectively.} \tag{C.15}$$

Hence, (i) - (vi) of Theorem 2 follows immediately from (C.10) -(C.15).

Part (b) : The proof is a simple application of the proof in Choi and Ahn (1993a).

Proof of Theorem 3 and Proof of Lemma 4.

These results are trivially obtained by applying the methods used in the proof of Theorem 1.

Proof of Lemma 5.

To prove Lemma 5, it is enough to show that we can transform the models to the form of $M(1)$ and $M(2)$ when there are multiple structural breaks. It is trivial to transform the model to $M(1)$ without the continuity restriction. Under the continuity restriction, we have the following q restrictions;

$$a_k^1 = a_k^2 + (a_{k+1}^2 - a_{k+1}^1)T_1 + \cdots + (a_\ell^2 - a_\ell^1)T_1^{\ell-k} \tag{C.16}$$

$$a_k^2 = a_k^3 + (a_{k+1}^3 - a_{k+1}^2)T_2 + \cdots + (a_\ell^3 - a_\ell^2)T_2^{\ell-k} \tag{C.17}$$

$$\vdots \tag{C.18}$$

$$a_k^q = a_k^{q+1} + (a_{k+1}^{q+1} - a_{k+1}^q)T_q + \cdots + (a_\ell^{q+1} - a_\ell^q)T_q^{\ell-k}. \tag{C.19}$$

Note that these q restrictions reduces the number of parameters to be estimated by q for each equation and a_k^i is expressed as follow:

$$a_k^i = a_k^{q+1} + \sum_{h=k+1}^{\ell} \sum_{j=i}^{q} (a_h^{j+1} - a_h^j) T_j^{h-k}. \tag{C.20}$$

Substituting a_k^i with the right hand side of (C.20) for $i = 1, \cdots, q$, into equation (4.44) and using the indicator functions $\eta_i = \sum_{j=1}^{q+1} \iota_j$, results in the desired equation.

Proof of Theorem 6.

The desired results follow immediately by applying continuous mapping theorem to Theorem 3.

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